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MULTIPLICATIVE ARITHMETIC FUNCTIONS

BY

K. G. RAMANATHAN, *University of Madras**

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1. Dr. R. Vaidyanathaswamy has proved† that every quadratic multiplicative arithmetic function satisfies a certain identity. In this paper, I obtain this identity by evaluating the Dirichlet's series $\sum_{n=1}^{\infty} f(Mn)/n^s$, where M is a positive integer. The main result proved is

$$\sum_{n=1}^{\infty} \frac{f(Mn)}{f(M)} \cdot n^{-s} = \prod_{p/M} \left(1 - \frac{B(p)\, f(p^{\alpha-1})\, p^{-s}}{f(p^{\alpha})} \right) \cdot \sum_{n=1}^{\infty} f(n) n^{-s},$$

where $f(n)$ is a quadratic function, and $M = p^{\alpha} \ldots$ We deduce from this, the following result :

$$\tau(MN) = \sum_{k=1}^{\infty} \tau(M/k)\tau(N/k)\mu(k)k^{11},$$

where $\tau(n)$ is Ramanujan's arithmetic function‡, and $\mu(k)$ is the Moebius function§ which vanishes when k is not quadratfrei, and which is equal to $(-1)^{\nu}$ when k is the product of ν distinct primes.

2. LEMMA 1. *If $f(n)$ is a multiplicative arithmetic function,*

$$f(m)\, f(n) = f(\delta)\, f(mn/\delta), \quad \delta = (m, n).$$

* I wish to express my thanks to Dr. R. Vaidyanathaswamy for his guidance in the preparation of this paper.

† *Trans. Amer. Math. Soc.* 3 (1931), 579-662.

‡ *Collected papers*, Cambridge (1927), p. 151.

§ Hardy and Wright : *Introduction to the Theory of Numbers*, p. 234.

For if $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$

and $n = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$

$$f(m)\, f(n) = \prod_{r=1}^{k} f(p_r^{\alpha_r})\, f(p_r^{\beta_r})$$

$$= \prod_{r=1}^{k} f(p_r^{A_r}) \prod_{r=1}^{k} f(p_r^{B_r}),$$

where A_r is min (α_r, β_r), B_r is max (α_r, β_r).

LEMMA 2. *If $m_1, m_2, \dots, m_\mu$ be positive integers, and g_r denotes the greatest common divisor (g.c.d.) of the l.c.m's of these integers taken r at a time, then*

$$f(m_1)\, f(m_2) \dots f(m_\mu) = f(g_1)\, f(g_2) \dots f(g_\mu).$$

This follows from Lemma 1. Also, since Lemma 1 is equivalent to the statement

$$f(Mx)\, f(My) = f(M)\, f(Mxy),$$

whenever x and y are co-prime integers, we have

LEMMA 3. *If $f(n)$ is any multiplicative function, so is $f(Mn)/f(M)$, where M is any positive integer.*

3. Denoting $f(Mn)/f(M)$ by $F_M(n)$, we shall now evaluate the Dirichlet's series $\sum_{n=1}^{\infty} F_M(n) n^{-s}$.

The generating series* for a prime base p is

$$1 + F_M(p)\, p^{-s} + F_M(p^2)\, p^{-2s} + \dots$$

If p be a prime not dividing M, the generating series for all such prime bases is

$$X_p(s) \equiv 1 + f(p)\, p^{-s} + f(p^2)\, p^{-2s} + \dots,$$

since $F_M(p^k) = f(p^k)$, $(p, M) = 1$. If p^* be a prime which divides $M = p^{*\alpha} \dots$, then since $F_M(p^{*k}) = f(p^{*\alpha+k})/f(p^{*\alpha})$, the generating series now becomes

$$Y_p(s) \equiv 1 + \frac{f(p^{\alpha+1})}{f(p^\alpha)}\, p^{-s} + \frac{f(p^{\alpha+2})}{f(p^\alpha)}\, p^{-2s} + \dots$$

Hence,

$$\sum_{n=1}^{\infty} F_M(n) n^{-s} = \Pi X_p(s)\, \Pi Y_{p^*}(s).$$

* R. Vaidyanathaswamy, *loc. cit.*

An arithmetic function $g(n)$ is *linear* if for every pair of integers m and n, we have $g(m)g(n)=g(mn)$. Let $f(n)$ be the composite* of the linear functions $G(n)$ and $g(n)$ so that

$$f(n) = \sum_{\delta\delta'=n} G(\delta)g(\delta').$$

Then

$$1+f(p)x+f(p^2)x^2+\ldots = [1-G(p)x]^{-1}\,[1-g(p)x]^{-1}$$
$$= \{1-f(p)x+B(p)x^2\}^{-1},$$

where $x=p^{-s}$, and $G(p)\,g(p)=f^2(p)-f(p^2)=B(p)$, so that $B(n)$ is also a linear function. Hence, the sum of the series $\Upsilon_{p^\alpha}(s)$, which is a recurring series of order two is

$$\Upsilon_p(s)=\left[1-\frac{B(p)f(p^{\alpha-1})x}{f(p^\alpha)}\right][1-f(p)x+B(p)x^2]^{-1}.$$

Hence,

$$\sum_{n=1}^{\infty} F_M(n)n^{-s} = \prod_{p\nmid M}[1-f(p)p^{-s}+B(p)p^{-2s}]^{-1}\prod_{p\mid M}\Upsilon_p(s)$$
$$=\prod_p[1-f(p)p^{-s}+B(p)p^{-2s}]^{-1}\prod_{p\mid M}\left[1-\frac{B(p)f(p^{\alpha-1})p^{-s}}{f(p^\alpha)}\right].$$

The two products on the right side are respectively equal to

$$\sum_{n=1}^{\infty} f(n)n^{-s} \text{ and } \sum_{\delta\mid M}\frac{\mu(\delta)\,B(\delta)\,f(M/\delta)\delta^{-s}}{f(M)}.$$

By the composition of Dirichlet's series, we thus finally obtain the identical relation satisfied by quadratic functions, viz.

$$f(MN)=\Sigma f(M/\delta)f(N/\delta)\mu(\delta)B(\delta),$$

where δ is a common factor of M and N.

4. Ramanujan's arithmetic function† $\tau(n)$ is given by

$$\sum_{n=1}^{\infty}\tau(n)x^n = x[(1-x)(1-x^2)\ldots]^{24}.$$

*R. Vaidyanathaswamy, *loc. cit.*

† *Collected papers*, p. 153.

Mordell* proved (what was conjectured by Ramanujan) that $\tau(n)$ is multiplicative, and that

$$\sum_{n=1}^{\infty} \tau(n) n^{-t} = \Pi \left(1 - \tau(p)\, p^{-t} + p^{11-2t} \right)^{-1}.$$

It follows that $\tau(n)$ is quadratic, and we deduce from § 3 that

$$\Sigma \tau(Mn) n^{-t} = \Sigma \tau(n) n^{-t} \cdot \prod_{p \mid M} [\tau(p^{\alpha}) - p^{11-t} \tau(p^{\alpha-1})],$$

and $$\tau(MN) = \sum_{\delta \mid M,\, N} \tau(M/\delta) \tau(N/\delta) \mu(\delta) \delta^{11}$$

$$= \sum_{\delta=1}^{\infty} \tau(M/\delta) \tau(N/\delta) \mu(\delta) \delta^{11}.$$

In a similar manner, since $d(n)$, the number of divivisors of n, is a quadratic multiplicative function, we obtain Ramanujan's result†

$$d(uv) = \sum_{n=1}^{\infty} d(u/n) d(v/n) \mu(n).$$

5. We shall next take $f(n)$ to be a totient‡, which is a rational function of degree (1, 1). When $(p; M) = 1$, we have

$$1 + f(p)x + f(p^2)x^2 + \ldots = \frac{1 - G(p)x}{1 - g(p)x},$$

where $x = p^{-s}$, and $G(p)$, $g(p)$ denote the linear functions $\{ f(p^2) - f^2(p) \}/(f(p)$ and $f^2(p) - f(p^2)$. When $(p, M) > 1$, we have

$$1 + \frac{f(p^{\alpha+1})}{f(p^{\alpha})} p^{-s} + \ldots = [1 - g(p)\, p^{-s}]^{-1}.$$

If $F_M(n) = f(Mn)/f(M)$ as before, then

$$\sum_{n=1}^{\infty} F_M(n) n^{-s} = \sum_{n=1}^{\infty} f(n) n^{-s} \cdot \prod_{p \mid M} (1 - G(p)\, p^{-s})^{-1}.$$

$$= \sum_{n=1}^{\infty} f(n) n^{-s}\, \Sigma G(m) m^{-s},$$

* *Proc. Camb. Phil. Soc.* (1917), 117-24.

† *Collected papers*, p. 134.

‡ R. Vaidyanathaswamy, *loc. cit.*

where m is a number formed entirely out of the prime factors of M. By Dirichlet multiplication, we obtain

$$f(Mn)/f(M) = \Sigma f(n/\delta)G(\delta),$$

the summation extending over the divisors of n. This could also be written as

$$f(M)f(n) = \sum_{\delta \mid n} G(\delta)f(Mn/\delta).$$

If $\phi(n)$ is Euler's function, $\zeta(s)$ is Riemann's zeta function, and $\phi_s(M) = M^s(1-p_1^{-s})(1-p_2^{-s})\ldots$, we have

$$\text{(i)} \quad \sum_{n=1}^{\infty} \phi(Mn)n^{-s} = \frac{\phi(M)M^s}{\phi_s(M)} \cdot \frac{\zeta(s-1)}{\zeta(s)}$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \mu(Mn)n^{-s} = \frac{\mu(M)M^s}{\phi_s(M)} \cdot \frac{1}{\zeta(s)}$$

$$\text{(iii)} \quad \sum_{\delta \mid m,\, n} \mu(\delta)\phi\left(\frac{mn}{\delta}\right) = \phi(m)\phi(n).$$

$$\text{(iv)} \quad \sum \mu(\delta)\mu\left(\frac{mn}{\delta}\right) = \mu(m)\mu(n).$$

6. The preceding sections enable us to express the Dirichlet's series $\Sigma f(mn)n^{-s}$, when $f(n)$ is multiplicative, as the product of $\Sigma f(n)n^{-s}$ and a finite Dirichlet's series. Thus the convergence or divergence of $\Sigma f(mn)n^{-s}$ is governed by that of $\Sigma f(n)n^{-s}$. Now it is well known that the series for $1/\zeta(s)$ converges for $s > 1$, and converges to zero when $s = 1$. Hence, if m is any positive integer, we have from § 5,

$$\mu(m) + \tfrac{1}{2}\mu(2m) + \tfrac{1}{3}\mu(3m) + \ldots = 0.$$

If $c_m(n)$ is Ramanujan's trigonometrical sum*

$$c_m(n) = \sum_{1 \leqslant \lambda < m} e^{\frac{2n\pi i\lambda}{m}}, \quad (\lambda, m) = 1,$$

it may be proved that

$$c_m(n) + \tfrac{1}{2}c_{2m}(n) + \tfrac{1}{3}c_{3m}(n) + \ldots - 0.$$

* *Collected papers*, p. 179.

7. If $f(n)$ is a quadratic function, or a totient, then

$$F(M, s) = \frac{\Sigma f(Mn)n^{-s}}{\Sigma f(n)n^{-s}}$$

is multiplicative in the argument M, so that if $(M, M') = 1$,

$$\frac{\Sigma f(Mn)n^{-s}}{\Sigma f(n)n^{-s}} \cdot \frac{\Sigma f(M'n)n^{-s}}{\Sigma f(n)n^{-s}} = \frac{\Sigma f(MM'n)n^{-s}}{\Sigma f(n)n^{-s}},$$

i.e. $\Sigma f(Mn)n^{-s}\, \Sigma f(M'n)n^{-s} = \Sigma f(MM'n)n^{-s} . \Sigma f(n)n^{-s}$.

This leads to the interesting result, that when $(M, M') = 1$,

$$\sum_{\delta\delta' = n} f(M\delta) f(M'\delta') = \sum_{\delta\delta' = n} f(MM'\delta) f(\delta').$$

OPERATIONAL CALCULUS AND HANKEL TRANSFORMS

BY

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[Received in revised form, 10 August 1943]

1. The object of this paper is to use operational calculus in the investigation of pairs of Hankel transforms. Two theorems have been established for the derivation of such a pair from functions whose operational images are known to be mutually related in a certain manner. Of the several pairs of Hankel transforms obtained by means of these theorems, results 1 to 11 are believed to be new.

2. THEOREM 1. *If*

$$f(p) \doteq h(x) \text{ and } \psi(p) = p^{1-\nu} f(1/p) \doteq g(x),$$

then the functions $x^{\frac{1}{2}-\nu} g(\frac{1}{2}x^2)$ *and* $x^{\frac{1}{2}-\nu} h(\frac{1}{2}x^2)$ *are a pair of* $\mathcal{J}_\nu$ *transforms, provided that the integrals*

$$\int_0^\infty x^{1-\nu} \mathcal{J}_\nu(xy) h(\tfrac{1}{2}x^2)\,dx \text{ and } \int_0^\infty x^{1-\nu} \mathcal{J}_\nu(xy) g(\tfrac{1}{2}x^2)\,dx$$

*converge.**

PROOF. Tricomi's† theorem states that if $f(p) \doteq h(x)$, then

$$p^{1-\nu} f(1/p) \doteq \int_0^\infty (x/s)^{\frac{1}{2}\nu} \mathcal{J}_\nu(2\sqrt{sx}) h(s)\,ds.$$

And by Lerch's theorem‡ each image-function represents

* The behaviours of the various functions for large and small values of the argument, required for the investigation of the convergence conditions, are shown in the Table at the end.

† Tricomi : *Rend. dei. Lincei*, 22 (1935), p. 564.

‡ Lerch : *Acta Math.* 27 (1903), pp. 339-51.

operationally a unique original function. Hence from the given operational relations we have

$$g(x) = \int_0^\infty (x/s)^{\frac{1}{2}\nu} J_\nu(2\sqrt{sx})h(s)\,ds. \tag{2.1}$$

Applying again Tricomi's theorem to the second operational relation, we have

$$p^{1-\nu}\psi(1/p) \doteqdot \int_0^\infty (x/s)^{\frac{1}{2}\nu} J_\nu(2\sqrt{sx})g(s)\,ds.$$

But $$p^{1-\nu}\psi(1/p) = f(p) \doteqdot h(x).$$

Hence by Lerch's theorem, we have

$$h(x) = \int_0^\infty (x/s)^{\frac{1}{2}\nu} J_\nu(2\sqrt{sx})g(s)\,ds. \tag{2.2}$$

Now putting $s = \frac{1}{2}t^2$ and $x = \frac{1}{2}y^2$ in (2.1) and (2.2), we arrive at the result if we multiply either side by $x^{\frac{1}{2}-\nu}$.

Cor. 1. *If* $f(p) \doteqdot h(x)$ *and* $p^{1-\nu} f(1/p) \doteqdot g(x)$, *then* $\phi(x) \equiv x^{\frac{1}{2}-\nu}\{g(\frac{1}{2}x^2\lambda) + \lambda^{-1+\nu}h(x^2/2\lambda)\}$ *is* R_ν.

If $g(x)$ and $h(x)$ are analytic functions, this result is valid by analytic continuation when the integral

$$\int_0^\infty x^{\frac{1}{2}} J_\nu(xy)\phi(x)\,dx$$

converges. In particular we have

Cor. 2. *If* $f(p) \doteqdot h(x)$ *and* $p^{1-\nu} f(1/p) \equiv f(p)$ *then** $x^{\frac{1}{2}-\nu}h(\frac{1}{2}x^2)$ *is* R_ν.

We have

$$f(p) = p\int_0^\infty e^{-px}h(x)\,dx = p\int_0^\infty e^{-\frac{1}{2}y^2p}\,y\,h(\tfrac{1}{2}y^2)\,dy \tag{2.3}$$

* In fact we obtain the more general function

$$x^{\frac{1}{2}-\nu}\{h(\tfrac{1}{2}x^2\lambda) + \lambda^{\nu-1}h(\tfrac{1}{2}x^2/\lambda)\}:$$

but such extension may always be made in an R_ν function of the type $x^{\frac{1}{2}-\nu}h(\frac{1}{2}x^2)$.

so that $p^{1-\nu}f(1/p)=p^{-\nu}\int_0^\infty \exp\left(-\frac{y^2}{2p}\right)y\,h(\tfrac{1}{2}y^2)\,dy$

$$=p\int_0^\infty y^{1-\nu}h(\tfrac{1}{2}y^2)\,dy\int_0^\infty J_\nu(xy)x^{\nu+1}e^{-\frac{1}{2}px^2}\,dx,\text{*}$$

$$=p\int_0^\infty x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}px^2}\,dx\int_0^\infty \sqrt{xy}\,y^{\frac{1}{2}-\nu}h(\tfrac{1}{2}y^2)J_\nu(xy)\,dy, \qquad (2.4)$$

on inversion, which we assume to be permissible.

Hence by subtraction of (2.4) from (2.3), we have

$$0=p\int_0^\infty e^{-\frac{1}{2}px^2}x^{\nu+\frac{1}{2}}\,dx\left[x^{\frac{1}{2}-\nu}h(\tfrac{1}{2}x^2)-\int_0^\infty \sqrt{xy}\,y^{\frac{1}{2}-\nu}h(\tfrac{1}{2}y^2)J_\nu(xy)\,dy\right].$$

Since the above integral is zero for general values of the parameter p which occurs in the integrand only as an exponent, it follows that

$$x^{\frac{1}{2}-\nu}h(\tfrac{1}{2}x^2)-\int_0^\infty \sqrt{xy}\,J_\nu(xy)\,y^{\frac{1}{2}-\nu}h(\tfrac{1}{2}y^2)\,dy=0,$$

which proves the result.

3. We shall now illustrate this theorem by three examples in which the pairs of J_ν transforms derived are those obtainable from known integrals.†

EXAMPLE 1. From a known integral‡ we easily deduce that if $R(m-k+\frac{1}{2})>0$

$$f(p)=p^{\frac{1}{2}-m}e^{\frac{1}{2}p}W_{k,m}(p)\doteq\frac{x^{m-k-\frac{1}{2}}(1+x)^{m+k-\frac{1}{2}}}{\Gamma(\frac{1}{2}+m-k)}=h(x). \quad (3.1)$$

Also from another integral due to Meijer§, we have

$$p^{1-(m-k-\frac{1}{2})}f(1/p)\equiv p^{k+1}\exp(1/2p)\,W_{k,m}(1/p)$$

$$\doteq\frac{2x^{-k-\frac{1}{2}}K_{2m}(2\sqrt{x})}{\Gamma(\frac{1}{2}+m-k)\,\Gamma(\frac{1}{2}-m-k)}=g(x),\ R(\pm m-k)>-\tfrac{1}{2}.$$

* G. N. Watson: *Bessel Functions* p. 394, § 13.3 (4) referred to as B. F.

† G. N. Watson: *loc. cit.*, § 13.45 (2), § 13.31 (1), § 13.6 (4, 5). See also E. C. Titchmarsh: *Theory of Fourier Integrals*, § 7.11.

‡ Whittaker and Watson: *Modern Analysis*, p. 340, referred to as M. A.

§ C. S. Meijer: *Proc. Roya' Acad. Amsterdam*, 37 (1934), p. 805.

Hence the theorem furnishes the J_ν transforms

$$x^{\nu-2m+\frac{1}{2}}K_{2m}(x\sqrt{2}),\ 2^{\nu-3m}\Gamma(\nu-2m+1)x^{\nu+\frac{1}{2}}(x^2+2)^{2m-\nu-1},$$
$$R(\tfrac{1}{2}\nu+\tfrac{3}{4})>R(2m)<R(\nu+1)>0.$$

EXAMPLE 2. If $f(p)\doteq h(x)$ and $\lambda>0$, we know that*

$$e^{-\lambda x}h(x)\doteq\{p/(p+\lambda)\}f(p+\lambda).$$

Applying this rule to the relation

$$p^{-n}\exp(-1/p)\doteq x^{\frac{1}{2}n}J_n(2\sqrt{x}),\ R(n)>-1,$$

we have

$$f(p)=\frac{p}{(p+\lambda)^{n+1}}\exp\left(-\frac{1}{p+\lambda}\right)\doteq x^{\frac{1}{2}n}e^{-\lambda x}J_n(2\sqrt{x})=h(x),$$
$$R(n)>-1.$$

From a similar operational relation we easily obtain

$$p^{-n}\exp\left(\frac{1}{\lambda^2 p}\right)\doteq(\lambda^2x)^{\frac{1}{2}n}I_n\left(\frac{2\sqrt{x}}{\lambda}\right).$$

Applying the rule stated above to this relation, we have

$$\frac{p}{(p+1/\lambda)^{n+1}}\exp\left\{\frac{1}{\lambda^2(p+1/\lambda)}\right\}\doteq\lambda^nx^{\frac{1}{2}n}\exp\left(-\frac{x}{\lambda}\right)I_n\left(\frac{2\sqrt{x}}{\lambda}\right),$$

that is,

$$p^{1-n}f\left(\frac{1}{p}\right)=\frac{p}{(p\lambda+1)^{n+1}}\exp\left(\frac{1}{\lambda(1+p\lambda)}-\frac{1}{\lambda}\right)$$
$$\doteq\frac{1}{\lambda}x^{\frac{1}{2}n}\exp\left(-\frac{1+x}{\lambda}\right)I_n\left(\frac{2\sqrt{x}}{\lambda}\right)=g(x).$$

Hence we get the pair of J_n transforms

$$\lambda\sqrt{x}\,e^{-\frac{1}{2}\lambda x^2}J_n(x\sqrt{2}),\ \sqrt{x}\exp\left(-\frac{x^2+2}{2\lambda}\right)I_n\left(\frac{x\sqrt{2}}{\lambda}\right),$$
$$R(n)>-1,\ \lambda>0.$$

EXAMPLE 3. Consider the operational relation†

$$h(x)=\frac{x^l}{(1+x^2)^n}\doteq\frac{\Gamma(l-2n+1)}{p^{l-2n}}\,{}_1F_2(n;\,n-\tfrac{1}{2}l,\,n-\tfrac{1}{2}l+\tfrac{1}{2};\,-\tfrac{1}{4}p^2)$$

* P. Humbert, *Le Calcul Symbolique*, pp. 27-31, This book will be referred to as C. S.

† H. C. Gupta: *On Operational Calculus* (a paper communicated for publication).

$$+\frac{1}{2\Gamma(n)}[\Gamma(\tfrac{1}{2}l+\tfrac{1}{2})\Gamma(n-\tfrac{1}{2}l-\tfrac{1}{2})\,p\,{}_1F_2(\tfrac{1}{2}l+\tfrac{1}{2};\tfrac{1}{2},\tfrac{3}{2}+\tfrac{1}{2}l-n;-\tfrac{1}{4}p^2)$$
$$-\Gamma(1+\tfrac{1}{2}l)\Gamma(n-\tfrac{1}{2}l-1)\,p^2\,{}_1F_2(1+\tfrac{1}{2}l;\tfrac{3}{2},2+\tfrac{1}{2}l-n;-\tfrac{1}{4}p^2)]=f(p).$$

The original of $p^{1-\nu}f(1/p)$ is easily obtained on termwise interpretation by means of the formula $p^{-n}\doteqdot x^n/\Gamma(n+1)$, $R(n)>-1$. And applying our theorem we obtain finally the $\mathcal{J}_\nu$ transforms

$$\frac{x^{2l+\frac{1}{2}-\nu}}{(x^4+4)^n}, \quad \frac{\Gamma(l-2n+1)}{\sqrt{2}\,\Gamma(\nu+2n-l)}\left(\frac{x}{2}\right)^{4n-2l+\nu-\frac{3}{2}}$$
$$\times\,{}_1F_4\left(\begin{matrix}n;\ n-\tfrac{1}{2}l,\ n-\tfrac{1}{2}l+\tfrac{1}{2},\\ \tfrac{1}{2}\nu+n-\tfrac{1}{2}l,\ \tfrac{1}{2}(\nu-l+1)+n;\end{matrix}\ -\tfrac{1}{64}x^4\right)+\frac{2^{l-\nu-2n-1}x^{\nu+\frac{1}{2}}}{\Gamma(n)}$$
$$\times\left\{\frac{\Gamma(\tfrac{1}{2}l+\tfrac{1}{2})\Gamma(n-\tfrac{1}{2}l-\tfrac{1}{2})}{\Gamma(\nu+1)}\,{}_1F_4\left(\begin{matrix}\tfrac{1}{2}l+\tfrac{1}{2};\ \tfrac{1}{2},\ \tfrac{1}{2}\nu+\tfrac{1}{2}\\ \tfrac{1}{2}\nu+1,\ \tfrac{3}{2}+\tfrac{1}{2}l-n;\end{matrix}\ -\tfrac{1}{64}x^4\right)\right.$$
$$\left.-\frac{\Gamma(1+\tfrac{1}{2}l)\Gamma(n-\tfrac{1}{2}l-1)}{2\Gamma(\nu+2)}x^2\,{}_1F_4\left(\begin{matrix}1+\tfrac{1}{2}l;\ \tfrac{3}{2},\ \tfrac{1}{2}\nu+1,\\ \tfrac{1}{2}\nu+\tfrac{3}{2},\ 2+\tfrac{1}{2}l-n;\end{matrix}\ -\tfrac{1}{64}x^4\right)\right\}$$
$$=F(x),\text{ say.}$$
$$R(2n+\tfrac{1}{2}\nu-\tfrac{1}{4})>R(l)>-1<R(\nu)<R(2l+\tfrac{3}{2}), \tag{3.2}$$

On using the formula

$$2\nu\mathcal{J}_\nu(x)K_\nu(x)={}_0F_3(\tfrac{1}{2},\ 1+\tfrac{1}{2}\nu,\ 1-\tfrac{1}{2}\nu;\ -\tfrac{1}{64}x^4)$$
$$+\frac{\nu x^2}{2(1-\nu^2)}\,{}_0F_3(\tfrac{3}{2},\ \tfrac{3}{2}+\tfrac{1}{2}\nu,\ \tfrac{3}{2}-\tfrac{1}{2}\nu;\ -\tfrac{1}{64}x^4)+\frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{x^2}{4}\right)^\nu$$
$$\times\,{}_0F_3(\tfrac{1}{2}\nu+\tfrac{1}{2},\ 1+\tfrac{1}{2}\nu,\ 1+\nu;\ -\tfrac{1}{64}x^4),$$

it may be seen that the hypergeometric functions occurring in (3.2) reduce to the product of two Bessel functions in the following cases: (i) $n=\nu+\frac{1}{2}$, $l=\nu$ and (ii) $n=\nu+\frac{1}{2}$, $l=\nu+1$, when we obtain the $\mathcal{J}_\nu$ transforms

$$\left(\frac{x}{x^4+4}\right)^{\nu+\frac{1}{2}}, \quad \frac{\sqrt{\pi}\,x^{\nu+\frac{1}{2}}}{2^{3\nu}\Gamma(\nu+\frac{1}{2})}\mathcal{J}_\nu(x)K_\nu(x), \quad R(\nu)>-\tfrac{1}{2};$$

and

$$\frac{x^{\nu+\frac{5}{2}}}{(x^4+4)^{\nu+\frac{1}{2}}}, \quad \frac{\sqrt{\pi}x^{\nu+\frac{1}{2}}}{2^{3\nu-1}\Gamma(\nu+\frac{1}{2})}\mathcal{J}_{\nu-1}(x)K_{\nu-1}(x),\ R(\nu)>\tfrac{1}{6}.$$

The former has been pointed out by Titchmarsh.*

* E. C. Titchmarsh: *loc. cit.*, p. 244.

4. We shall now obtain Hankel transforms of order ν such that the functions in each pair are of the same type and are also self-reciprocal in the Hankel transform of orders adding up to 2ν. Their sums (or differences) are therefore generalizations of the single R_ν (or $-R_\nu$) functions which result when the two transforms become identical and which are also obtainable direct by means of Cor. 2.

Result 1. From an integral due to Erdelyi* we get

$$f(p) = \frac{p^{k-m+\frac{1}{2}}}{(1+p)^{k+m+\frac{1}{2}}} \doteqdot \frac{x^{m-\frac{1}{2}}}{\Gamma(2m+1)} e^{-\frac{1}{2}x} M_{k,m}(x)$$

$$= \frac{x^{2m}}{\Gamma(2m+1)} e^{-x} {}_1F_1(\tfrac{1}{2}+m-k;\ 2m+1;\ x) = h(x),\ R(m) > -\tfrac{1}{2}.$$

Also

$$p^{1-\nu} f\left(\frac{1}{p}\right) = \frac{p^{2m+1-\nu}}{(1+p)^{k+m+\frac{1}{2}}} \doteqdot \frac{x^{\mu-\frac{1}{2}}}{\Gamma(2\mu+1)} e^{-\frac{1}{2}x} M_{\mu+2m-\nu+\frac{1}{2},\mu}(x),$$

where $2\mu = k+\nu-m-\frac{1}{2}$ and $R(\mu) > -\frac{1}{2}$.

Hence we get the pair of $\mathfrak{J}_\nu$ transforms

$$\frac{x^{4m+\frac{1}{2}-\nu}}{2^{2m}\Gamma(2m+1)} e^{-\frac{1}{2}x^2} {}_1F_1\left(\begin{matrix}\frac{1}{2}+m-k; \\ 2m+1;\end{matrix} \tfrac{1}{2}x^2\right),$$

$$\frac{x^{2k+\nu-2m-\frac{1}{2}}}{\Gamma(k+\nu-m+\frac{1}{2})} e^{-\frac{1}{2}x^2} {}_1F_1\left(\begin{matrix}\nu-2m; \\ k+\nu-m+\frac{1}{2};\end{matrix} \tfrac{1}{2}x^2\right), \quad (4.1)$$

$$0 < R(2m+1) > R(\tfrac{1}{2}\nu+\tfrac{1}{4}) > R(m-k) < R(\nu+\tfrac{1}{2}).$$

On taking $k+\nu=3m+\frac{1}{2}$ the two ${}_1F_1$ functions become identical and we obtain the R_ν function due to B. Mohan†, viz. $x^{4m+\frac{1}{2}-\nu} e^{-\frac{1}{2}x^2}\ {}_1F_1\ (\nu-2m;\ 2m+1;\ \frac{1}{2}x^2)$.

It is now obvious that of the two functions in (4.1) the former is $R_{3m-k+\frac{1}{2}}$ and the latter $R_{k+2\nu-3m-\frac{1}{2}}$.

Result 2. If $f_1(p) \doteqdot h_1(x)$ and $f_2(p) \doteqdot h_2(x)$, then by a well-known rule

$$p^{-1} f_1(p) f_2(p) \doteqdot \int_0^x h_1(y) h_2(x-y) dy.$$

* A. Erdelyi: *Math. Zeit.* 42 (1936), p. 131.

† B. Mohan: *Quart. Jour. of Math.* 10 (1939), p. 256.

Applying this rule to the operational relations*

$$f_1(p)=\Gamma(n+1)p^{-n} \doteqdot x^n=h_1(x),\ n>-1$$

and $$f_2(p)=p/(p^2+1) \doteqdot \sin x=h_2(x)$$

we deduce that

$$f(p)=\frac{\Gamma(n+1)}{p^n(p^2+1)} \doteqdot \int_0^x y^n \sin(x-y)\,dy=h(x)$$

$$=\frac{x^n}{n+1} \sin x\ {}_1F_2\left(\frac{n+1}{2};\frac{1}{2},\frac{n+3}{2};-\frac{x^2}{4}\right)-\frac{x^{n+2}}{n+2}\cos x$$

$$\times {}_1F_2\left(\frac{2+n}{2};\frac{3}{2},\frac{4+n}{2};-\frac{x^2}{4}\right)=\psi(x,n),\ \text{say}.$$

Since

$$p^{1-\nu}f\left(\frac{1}{p}\right)=\frac{\Gamma(n+1)}{p^{\nu-n-3}(1+p^2)} \doteqdot \frac{\Gamma(n+1)}{\Gamma(\nu-n-2)}\psi(x,\nu-n-3)$$

we obtain the pair of J_ν transforms

$$\Gamma(\nu-n-2)\,x^{\frac{1}{2}-\nu}\psi(\tfrac{1}{2}x^2,n),\ \Gamma(n+1)x^{\frac{1}{2}-\nu}\psi(\tfrac{1}{2}x^2,\nu-n-3), \quad (4.2)$$

$$-1<R(\nu)-\tfrac{3}{2}>2R(n),\ R(\nu)>R(n)>-3.$$

On taking $\nu=2n+3$, we get the single R_{2n+3} function

$$x^{-\frac{5}{2}-2n}\psi(\tfrac{1}{2}x^2,n),\quad R(n)>-\tfrac{5}{4}. \qquad (4.3)$$

It is now apparent that of the transforms (4.2), the former is R_{2n+3} and the latter is $R_{2\nu-2n-3}$.

On taking $n=-1$ in (4.3) we obtain the familiar R_1 function $x^{-\frac{1}{2}}\sin(\frac{1}{2}x^2)$.

If we take $n=0$, and use the formulæ

$x\ {}_0F_1(\frac{3}{2},-\frac{1}{4}x^2)=\sin x$ and $x^2\ {}_1F_2(1;\frac{3}{2},2;-\frac{1}{4}x^2)=4\sin^2(\frac{1}{2}x)$

we get an R_3 function $x^{-\frac{5}{2}}\sin^2(\frac{1}{2}x^2)$.

5. THEOREM 2. *If $f(p)\doteqdot h(x)$ and $g(p)\doteqdot x^{\nu-1}h(1/x)$, then the functions*

$$x^{\nu-\frac{3}{2}}f(\tfrac{1}{2}x^2) \text{ and } x^{\nu-\frac{3}{2}}g(\tfrac{1}{2}x^2)$$

are a pair of J_ν transforms, provided that the integrals

$\int_0^\infty x^{\nu-1}J_\nu(xy\ f(\frac{1}{2}x^2)\,dx$ and $\int_0^\infty x^{\nu-1}J_\nu(xy)g(\frac{1}{2}x^2)\,dx$ *converge.*

* C. S. pp. 27-31.

The proof is similar to that of Theorem 1, except that instead of Tricomi's theorem we use its parallel,* viz. If $f(p) \doteq h(x)$, then

$$x^{\nu-1} h\left(\frac{1}{x}\right) \doteq p^{1-\frac{1}{2}\nu} \int_0^\infty t^{\frac{1}{2}\nu-1} J_\nu(2\sqrt{tp}) f(t) dt.$$

Cor. *If* $f(p) \doteq h(x)$ *and* $x^{\nu-1}h(1/x) \doteq g(p)$, *then the function* $\phi(x) = x^{\nu-\frac{3}{2}}\{ f(\frac{1}{2}x^2\lambda) + \lambda^{1-\nu} g(\frac{1}{2}x^2/\lambda)\}$ *is* R_ν, *if*

$$\int_0^\infty x^{\frac{1}{2}} J_\nu(xy)\phi(x)dx \text{ converges.}$$

In particular if $f(p) \doteq h(x)$ and $x^{\nu-1}h(1/x) = h(x)$ then $x^{\nu-\frac{3}{2}} f(\frac{1}{2}x^2)$† is R_ν. A direct proof on the lines of that given in Th. 1 Cor. 2, may be supplied employing the same integral as used therein. We however proceed to the application of the theorem to obtain new J_ν transforms.

Result 3. Consider the operational relation‡

$$h(x) \equiv x^{-1} J_\mu(1/x) \doteq 2p J_\mu(2\sqrt{p})\, K_\mu(2\sqrt{p}) \equiv f(p).$$

Also we have§

$$x^{\lambda-1} h\left(\frac{1}{x}\right) \equiv x^\lambda J_\mu(x) \doteq \frac{\Gamma(\lambda+\mu+1)}{2^\lambda \Gamma(\mu+1)} \frac{1}{p^{\lambda+\mu}}$$

$$\times {}_2F_1\left(\begin{matrix} \frac{1}{2}(\lambda+\mu+1), \frac{1}{2}(\lambda+\mu+2); \\ \mu+1; \end{matrix} -\frac{1}{p^2} \right) \equiv g(p), \; R(\lambda+\mu) > -1.$$

Theorem 2 therefore yields the J_λ transforms

$$x^{\lambda+\frac{1}{2}} J_\mu(x\sqrt{2}) K_\mu(x\sqrt{2}),$$

$$\frac{\Gamma(\lambda+\mu+1)2^\mu}{\Gamma(\mu+1)x^{\lambda+2\mu+\frac{3}{2}}} {}_2F_1\left(\begin{matrix} \frac{1}{2}(\lambda+\mu+1), \frac{1}{2}(\lambda+\mu+2); \\ \mu+1; \end{matrix} -\frac{4}{x^4} \right),$$

where $R(\frac{1}{2}\lambda - \frac{1}{4}) > -R(\mu+1) < R(\lambda) > -1$.

The function ${}_2F_1$ occurring above may be summed up when $\lambda = \mu$ or $\mu+1$ by means of the formula ${}_1F_0(\alpha, -z) =$

* H. C. Gupta : *loc. cit.*

† A remark similar to that in the corollary of the previous theorem applies here as well.

‡ S. C. Mitra: *Bull. Calcutta Math. Soc.* 25 (1933), 81-98.

§ E. T. Copson: *Proc. Lond. Math. Soc.* (3), 33 (1933), 145-53.

$(1+z)^a$; and we obtain the transforms derived as particular cases in Ex. 3. The function ${}_2F_1$ can also be summed up for $\lambda=0$ by the formula

$$ {}_2F_1\left(\frac{\mu+1}{2}, \frac{\mu+2}{2}; \mu+1; -x^2\right) = \frac{1}{\sqrt{(1+x^2)}}\left\{\frac{\sqrt{(1+x^2)}-1}{\frac{1}{2}x^2}\right\}^{\mu}, $$

when we obtain the J_0 transforms

$$ x^{\frac{1}{2}} J_\mu(x\sqrt{2})K_\mu(x\sqrt{2}); \frac{\sqrt{x}}{\sqrt{(x^4+4)}}\left\{\sqrt{(x^4+4)}-x^2\right\}^{\mu}, \quad R(\mu)>-\tfrac{3}{4}. $$

Result 4. Applying the theorem to the operational relation of Ex. 1 and proceeding as above we get the J_ν transforms

$$ (x/\sqrt{2})^{\nu-2m-\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,m}(\tfrac{1}{2}x^2); $$

$$ \frac{\Gamma(\nu-2m+1)}{\Gamma(\frac{1}{2}+m-k)}\left(\frac{x}{\sqrt{2}}\right)^{m-k-1} e^{\frac{1}{4}x^2} W_{\frac{1}{2}(3m+k-\frac{1}{2}-\nu),\, \frac{1}{2}(k+\nu-m+\frac{1}{2})}(\tfrac{1}{2}x^2) $$

where $\quad -\frac{1}{2} < R(m-k) > R(\frac{1}{2}\nu-\frac{1}{4}) > R(2m-1) < R(\nu) > -1.$

The two Whittaker functions become identical when $k=3m-\nu-\frac{1}{2}$; and we obtain the R_ν function due to Bailey*, viz. $\quad x^{\nu-2m-\frac{1}{2}} e^{\frac{1}{4}x^2} W_{3m-\nu-\frac{1}{2},\, m}(\frac{1}{2}x^2)$,

$$ R(\tfrac{1}{2}\nu-\tfrac{1}{4}) > R(2m-1) < R(\nu) > -1. $$

Result 5. Take the operational relation†

$$ h(x) \equiv x^n \log x \doteqdot \Gamma(n+1)p^{-n}\{\psi(n+1)-\log p\} \equiv f(p) $$

and the following deducible therefrom

$$ x^{\nu-1}h(1/x) \equiv -x^{\nu-n-1}\log x $$
$$ \doteqdot \Gamma(\nu-n)p^{n+1-\nu}\{\log p-\psi(\nu-n)\} \equiv g(p). $$

Theorem 2 then furnishes the J_ν transforms

$$ 2^n\Gamma(n+1)x^{\nu-2n-\frac{3}{2}}\{\psi(n+1)-\log(\tfrac{1}{2}x^2)\}, $$
$$ \Gamma(\nu-n)2^{\nu-n-1}x^{2n+\frac{1}{2}-\nu}\{\log(\tfrac{1}{2}x^2)-\psi(\nu-n)\}, $$
$$ R(\nu+1) > R(2n+\tfrac{3}{2}) > R(\nu) > R(n) > -1. $$

* W. N. Bailey: *Jour. Lond. Math. Soc.* 5 (1930), pp. 258-65.

† C. S. pp. 27-31.

The two transforms add up to zero if $2n=\nu-1$ and we get the $-R_\nu$ function $x^{-\frac{1}{2}}\{\psi(\frac{1}{2}\nu+\frac{1}{2})-\log(\frac{1}{2}x^2)\}$, $R(\nu)>-1$.

If we introduce a parameter and subtract the functions so obtained from the original ones, we obtain the $\mathfrak{J}_\nu$ transforms $2^n\Gamma(n+1)x^{\nu-\frac{3}{2}-2n}$, $\Gamma(\nu-n)2^{\nu-n-1}x^{2n+\frac{1}{2}-\nu}$,

$$R(\nu+1)>R(2n+\tfrac{3}{2})>R(\nu)>R(n)>-1.$$

6. In this section we shall apply the two theorems of this paper to operational relations* involving logarithmic and inverse trigonometric functions given below; and obtain $\mathfrak{J}_1$ and $\mathfrak{J}_0$ transforms. It is interesting to note that the self-reciprocal or skew self-reciprocal functions formed from some of these transforms are capable of expression in a closed form.

The operational relations are:—

(i) $-\gamma-\log p\doteqdot\log x$

(ii) $\cos p\ \mathrm{ci}(p)+\sin p\ \mathrm{si}(p)\doteqdot-\frac{1}{2}\log(1+x^2)$

(iii) $\arctan p\doteqdot-\mathrm{si}(x)$

(iv) $\sin p\ \mathrm{ci}(p)-\cos p\ \mathrm{si}(p)\doteqdot\arctan x$

(v) $-\log(1+p^2)\doteqdot2\ \mathrm{ci}(x)$

(vi) $p^{-1}\log(1+p^2)\doteqdot-2[x\ \mathrm{ci}(x)-\sin x]$

(vii) $(\gamma+\log p)^2+\frac{1}{6}\pi^2\doteqdot\log^2 x$

(viii) $p\log(1+1/p^2)\doteqdot4x^{-1}\sin^2(\frac{1}{2}x)$,

where $\mathrm{si}(x)$ and $\mathrm{ci}(x)$ denote respectively the integrals

$$\int_\infty^x\frac{\sin x\,dx}{x}\quad\text{and}\quad\int_\infty^x\frac{\cos x}{x}\,dx$$

and γ is Euler's constant.

Result 6. Taking $f(p)\equiv(\gamma+\log p)^2+\frac{1}{6}\pi^2\doteqdot\log^2 x\equiv h(x)$ so that $f(1/p)\equiv(\gamma-\log p)^2+\frac{1}{6}\pi^2\doteqdot\log^2 x+4\gamma^2+4\gamma\log x\equiv g(x)$ we derive by Theorem 1 the $\mathfrak{J}_\nu$ transforms

$$x^{-\frac{1}{2}}\log^2(\tfrac{1}{2}x^2),\ x^{-\frac{1}{2}}\{\log^2(\tfrac{1}{2}x^2)+4\gamma(\gamma+\log\tfrac{1}{2}x^2)\}.$$

* P. Humbert: *Bull. Soc. Math. de France*, 65 (1937) pp. 1-13.

Their difference gives the $-R_1$ function $x^{-\frac{1}{2}}(\gamma+\log \frac{1}{2}x^2)$.

If we introduce two parameters λ and μ and take the differences of the corresponding functions, and note that $x^{-\frac{1}{2}}$ is R_ν we obtain, after simplification, the $\mathcal{J}_\nu$ transforms

$$x^{-\frac{1}{2}} \log(x/\sqrt{2})\,; -x^{-\frac{1}{2}}\, [\log(x/\sqrt{2})+\gamma].$$

Result 7. From (v) and (i) we obtain the $\mathcal{J}_1$ transforms $x^{-\frac{1}{2}}\mathrm{ci}(\frac{1}{2}x^2)$, $x^{-\frac{1}{2}}[-\gamma-\log(\frac{1}{2}x^2)+\mathrm{ci}(\frac{1}{2}x^2)]$.

Result 8. From (iii) we get the $\mathcal{J}_1$ transforms

$$-x^{-\frac{1}{2}}\, \mathrm{si}(\tfrac{1}{2}x^2),\ x^{-\frac{1}{2}}\left\{ \tfrac{1}{2}\pi+\mathrm{si}(\tfrac{1}{2}x^2) \right\}.$$

Their sum gives the familiar $\mathcal{J}_1$ function $x^{-\frac{1}{2}}$.

Result 9. Using (viii) and (v) we get the $\mathcal{J}_0$ transforms $8x^{-\frac{3}{2}}\sin^2(\frac{1}{4}x^2)$, $-2\sqrt{x}\,\mathrm{ci}(\frac{1}{2}x^2)$.

Result 10. Employing (viii) and (vi) we obtain the $\mathcal{J}_1$ transforms $8x^{-\frac{5}{2}}\sin^2(\frac{1}{2}x^2)$, $-2x^{-\frac{1}{2}}[\frac{1}{2}x^2\,\mathrm{ci}(\frac{1}{2}x^2)-\sin(\frac{1}{2}x^2)]$.

Result 11. Taking (iv), we obtain the $\mathcal{J}_1$ transforms

$$x^{-\frac{1}{2}}[\sin\,(\tfrac{1}{2}x^2)\ \mathrm{ci}\,(\tfrac{1}{2}x^2)-\cos(\tfrac{1}{2}x^2)\ \mathrm{si}\,(\tfrac{1}{2}x^2)],$$

$$x^{-\frac{1}{2}}[\tfrac{1}{2}\pi-\sin\,(\tfrac{1}{2}x^2)\ \mathrm{ci}(\tfrac{1}{2}x^2)+\cos(\tfrac{1}{2}x^2)\ \mathrm{si}(\tfrac{1}{2}x^2)].$$

Result 12. Lastly from (ii) we have the $\mathcal{J}_1$ transforms

$$x^{-\frac{1}{2}}[\cos(\tfrac{1}{2}x^2)\ \mathrm{ci}(\tfrac{1}{2}x^2)+\sin(\tfrac{1}{2}x^2)\ \mathrm{si}(\tfrac{1}{2}x^2)],$$

$$x^{-\frac{1}{2}}[\cos(\tfrac{1}{2}x^2)\ \mathrm{ci}(\tfrac{1}{2}x^2)+\sin(\tfrac{1}{2}x^2)\ \mathrm{si}(\tfrac{1}{2}x^2)-\gamma-\log(\tfrac{1}{2}x^2)].$$

In conclusion the author wishes to express his thanks to Dr. R. S. Varma, and to the referee for their valuable suggestions for the improvement of the paper.

TABLE.

[Notation : $f(x)=O\{\phi_1(x)\}$ as $x\to 0$ and $f(x)=O\{\phi_2(x)\}$ as $x\to\infty$, a and b stand for some numerical constants different in each case; ε is an arbitrarily small positive number.]

$f(x)$	$\phi_1(x)$	$\phi_2(x)$	Ref.
$I_\nu(x)$	x^ν	$x^{-\frac{1}{2}}e^x$	B.F. 203 (2).
$J_\nu(x)$	x^ν	$x^{-\frac{1}{2}}\cos(x+a)$	,, 199 (1).
$K_\nu(x)$	$ax^{-\nu}+bx^\nu$	$x^{-\frac{1}{2}}e^{-x}$	,, 202 (1).
$W_{k,m}(x)$	$ax^{\frac{1}{2}+m}+bx^{\frac{1}{2}-m}$	$x^k e^{-\frac{1}{2}x}$	M.A. 342.
$\log x$	$x^{-\varepsilon}$	x^{ε}	
ci(x)*	$x^{-\varepsilon}$†	$x^{-1}\sin x$	C.V. 231 Ex. 22.
si(x)	1	$x^{-1}\cos x$	,,
${}_1F_1(\alpha;\rho;-x)$	1	$x^{-\alpha}$	,, 265
${}_1F_1(\alpha;\rho;x)$	1	$x^{\alpha-\rho}e^x$	,,
${}_1F_2(\alpha;\lambda,\rho;+x)$‡	1	$ax^{\frac{1}{4}+\frac{1}{2}(\alpha-\lambda-\rho)}+bx^{-\alpha}$	
${}_2F_1(\alpha,\beta;\rho;-x)$	1	$ax^{-\alpha}+bx^{-\beta}$	,, 257
$F(x)$ [Ex. 3, (3.2)]	$ax^{\nu+\frac{1}{2}}$§ $bx^{4n-2l+\nu-\frac{3}{2}}$	$x^{\nu-2l-\frac{3}{2}}$‡	

*E.T. Copson: *Functions of a complex variable* (Oxford, 1937), 231 referred to as C.V.

† Since $\int_0^x (\cos t - e^{-t})dt/t = 0$, therefore

$$-\int_x^\infty t^{-1}\cos t\,dt = -\int_x^\infty t^{-1}e^{-t}\,dt + \int_0^x t^{-1}(\cos t - e^{-t})\,dt,$$

that is, $\mathrm{ci}(x) = [-e^{-t}\log t]_x^\infty - \int_x^\infty e^{-t}\log t\,dt + \int_0^x (1-t+\dots)dt$

$$= e^{-x}\log x + J, \text{ where } J = o(1) \text{ as } x \to 0.$$

‡ E.M. Wright: *Proc. Lond. Math. Soc.* 42 (1940), 389.

§ Deduced by representing $F(x)$ by Barnes contour integral

$$Ax^{\nu+\frac{1}{2}}\int_{c-i\infty}^{c+i\infty}\Gamma(-s)\Gamma(n-\tfrac{1}{2}l-\tfrac{1}{2}-\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}l+\tfrac{1}{2}+\tfrac{1}{2}s)(-\tfrac{1}{2}x^2)^s ds/\Gamma(1+\nu+s),$$

(where A is a constant independent of x and $c>0$) and carrying out the usual change of contour.

FOURIER ANSATZ AND NON-LINEAR PARABOLIC EQUATIONS*

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INTRODUCTION.

The purpose of this memoir is to solve the non-linear parabolic equation

$$\triangle u-\frac{\partial u}{\partial t}=\sum_{i=1}^{m}\frac{\partial^2 u}{\partial x_i^2}-\frac{\partial u}{\partial t}=f\left(x_1,\ldots,x_m,t,u,\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_m}\right) \quad (1)$$

for the boundary conditions

$$u(x_1, x_2,\ldots, x_m,t)=0 \text{ on } \Gamma,\ t>0 \quad (2)$$

where Γ is the boundary of a bounded closed domain G and

$$u(x_1,\ldots, x_m,0)=u_0(x_1,\ldots, x_m). \quad (3)$$

We shall obtain a solution in the form

$$u(x_1, x_2,\ldots, x_m, t)=\sum_{1}^{\infty}u_n(t)\omega_n(x_1,\ldots, x_m), \quad (4)$$

where $\omega_n(x_1,\ldots, x_m)$ are the eigenfunctions of

$$\triangle\omega+\mu\omega=0 \quad (5)$$

$$\omega(x_1,\ldots, x_m)=0 \text{ on } \Gamma. \quad (6)$$

If $m=1$, (1) becomes

$$\delta u=\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=f\left(x, t, u, \frac{\partial u}{\partial x}\right) \quad (7)$$

and the problem reduces to that of obtaining a solution in the form of a Fourier series. We have studied elsewhere†

* This memoir formed part of a thesis approved in 1941 for the D.Sc. of the Madras University.

† "Studies in Fourier Ansatz and Parabolic Equations," *Journal of the Madras University.* Vol. 14 (1942) pp. 73-142, and continued in *Jour. Ind. Math. Soc.* Vol. 6 (1942) pp. 41-50. These two papers together will be referred to as P. 1.

this problem in detail, and, as can be seen from that, the theory of parabolic equations when $m=1$ is particularly simple on account of the well-known results in the theory of Fourier series. But when $m=2$, we have to develop afresh a theory of "*Fourier series*" of eigenfunctions of (5) and obtain results analogous to those in the theory of trigonometrical series. Some results have already been obtained by the author in six papers "*On expansion in eigenfunctions of boundary-value problems.*"* The results of the first of these papers and a few more are necessary for our present study. Part I contains proofs of these results, while Part II is concerned with the linear homogeneous equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial t}=0 \tag{8}$$

and Parts III and IV are devoted to the discussion of the equations

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial t}=f(x, y, t) \tag{9}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial t}=f\left(x, y, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \tag{10}$$

It is of interest to note that we have made use of known properties of Banach spaces and the theory of transformations therein. The new notion of summability —Bessel summability—which the author has defined elsewhere is made use of in Part I.

It will be apparent that the discussion can be extended to the case $m>2$ as well.

I.—ON EXPANSION IN EIGENFUNCTIONS OF BOUNDARY VALUE PROBLEMS.

§1. In the two-dimensional domain let G denote a bounded domain whose boundary Γ is composed of a finite number of regular curves. Then the equation

$$\Delta\omega+\mu\omega=\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}+\mu\omega=0,\ \omega(x, y)=0 \text{ on } \Gamma \tag{11}$$

* See *Proc. Ind. Ac. Sci.* 12 (1940), 462; *Jour. Ind. Math. Soc.* 5 (1941), 103; 6 (1942) 153; 7(1943), 17 and 89; *Bull. Cal. Math. Soc.* 35 (1943), 81. These papers will be referred to as E.1, E.2, etc.

has an infinite number of normal orthogonal eigenfunctions $\omega_n(x, y)$ with corresponding positive eigenvalues μ_n arranged in increasing order of magnitude. We shall make use of the following properties of eigenfunctions and eigenvalues.*

$$\lim_{n \to \infty} \frac{\mu_n}{n} = l > 0 \tag{12}$$

so that $\Sigma \mu_n^{-\gamma}$ is convergent if $\gamma > 1$

and
$$\omega_n(x, y) = O(\mu_n) \tag{13}$$

$$\left.\begin{matrix} \dfrac{\partial \omega_n}{\partial x} \\ \dfrac{\partial \omega_n}{\partial y} \end{matrix}\right\} = O\left\{ \mu_n (\log \mu_n)^{\frac{1}{2}} \right\} \tag{14}$$

uniformly in $G+\Gamma$. Also if $\omega(x, y)$ is a solution of (11)

$$\omega(x, y)\, J_0(\mu^{\frac{1}{2}}\rho) = \frac{1}{2\pi} \int_0^{2\pi} \omega(x+\rho\cos\theta, y+\rho\sin\theta)\, d\theta \tag{15}$$

so that

$$2\,\omega(x, y)\, \frac{J_1(\mu^{\frac{1}{2}}\rho)}{\mu^{\frac{1}{2}}\rho} = \frac{1}{\pi\rho^2} \iint_{c\rho} \omega(\xi, \eta)\, d\xi\, d\eta, \tag{16}$$

where c_ρ is a circular region of radius ρ with centre (x, y) and $J_\mu(t)$ is a Bessel function of order μ.

§ 1.1 We shall further consider the following classes of functions defined in $G+\Gamma$, which form Banach Spaces with suitable norms.

(i) L^p, $p \geqslant 1$: the space of measurable functions $f(x, y)$ the pth powers of whose moduli are integrable in G.

(ii) B: the space of bounded measurable functions.

(iii) C^*: the space of functions $f(x, y)$ which are continuous in $G+\Gamma$ vanishing on Γ.

(iv) H_α: the subspace of C^* whose elements $f(x, y)$ satisfy the condition

$$|f(x+h, y+k) - f(x, y)| \leqslant K(h^2+k^2)^{\alpha/2}, \quad 0 < \alpha \leqslant 1. \tag{17}$$

*Cf. Courant and Hilbert: *Methoden der Mathematische Physik*, Bd. I.

(v) H_α^p: the subspace of L^p whose elements $f(x, y)$ satisfy the condition

$$\left\{\iint_G |f(x+h, y+k)-f(x,y)|^p dx\,dy\right\}^{\frac{1}{p}} \leqslant K(h^2+k^2)^{\alpha/2}, 0<\alpha\leqslant 1. \tag{18}$$

Since $\omega_n(x, y)$ form a complete system of normal orthogonal functions, we have the following Parseval's formula:

If $f(x, y) \varepsilon L^2$ and $g(x, y) \varepsilon L^2$ in G, and if

$$f(x, y) \sim \Sigma c_n \omega_n, \quad c_n = \iint f\,\omega_n dxdy \tag{19}$$

$$g(x, y \sim \Sigma d_n \omega_n, \; d_n = \iint g\omega_n dxdy \tag{20}$$

then

$$\iint fg dxdy = \Sigma c_n d_n. \tag{21}$$

§1.2. As a consequence of (21) we have the following lemmas.

Lemma 1. *If $f(x, y) \varepsilon L^2$ and if $f(x, y) \sim \Sigma c_n \omega_n$ then the series $\Sigma c_n \alpha_\mu(\lambda_n \rho)\omega_n(x, y)$ is absolutely convergent for all $\rho > 0$ provided $\mu > \frac{5}{2}$ and if (x, y) is interior to G, $f_\mu(\rho; x, y) = \sum_1^\infty c_n \omega_n \alpha_\mu(\lambda_n\rho) \to f(x, y)$, as $\rho \to 0$, at a point of continuity of $f(x, y)$.*

Proof. Using the notation

$$\lambda_n = \mu_n^{\frac{1}{2}} \text{ and } \alpha_\mu(x) = J_\mu(x)/2^\mu \Gamma(\mu+1)\, x^\mu, \tag{22}$$

since $f \varepsilon L^2, \Sigma c_n^2 < \infty$, we have from (12) and (13),

$$\Sigma |c_n \omega_n \alpha_\mu(\lambda_n \rho)| \leqslant \left\{ \Sigma c_n^2 \; \Sigma \omega_n^2 \alpha_\mu^2(\lambda_n\rho)\right\}^{\frac{1}{2}}$$

$$= \left\{\sum O\left(\frac{\mu_n^2}{\lambda_n^{2\mu+1}}\right)\right\}^{\frac{1}{2}} = \left\{\sum O\left(\frac{1}{n^{\mu-\frac{3}{2}}}\right)\right\}^{\frac{1}{2}} = O(1) \tag{23}$$

provided $\mu > \frac{5}{2}$. Taking

$$g(\xi, \eta) = \frac{\mu}{\pi r^2}\left(1-\frac{t^2}{r^2}\right)^{\mu-1}, \; t \leqslant r$$

$$= 0, \; t \geqslant r, \tag{24}$$

where $(\xi-x)^2+(\eta-y)^2=t^2$, and applying (21) we observe that

$$\iint_G fg\,d\xi d\eta = \frac{\mu}{\pi r^2}\iint_{C_r}\left(1-\frac{t^2}{r^2}\right)^{\mu-1} f(\xi,\eta)\,d\xi d\eta$$

$$=\Sigma c_n \omega_n \alpha_\mu(\lambda_n\rho)=f_\mu(\rho;x,y)\rightarrow f(x,y) \text{ as } \rho\rightarrow 0 \quad (25)$$

if $f(x,y)$ is continuous.

LEMMA 2. *If $f(x,y)\ \varepsilon\ L$ and $g(x,y)\ \varepsilon\ C^*$, then $\Sigma c_n d_n$ is summable (J_μ,λ) to the sum $\iint fg\,dxdy$ for $\mu>\frac{5}{2}$.*

PROOF. By Lemma 1,

$$g_\mu(\rho;x,y)=\Sigma d_n\alpha_\mu(\lambda_n\rho)\omega_n \quad (26)$$

the series on the right being absolutely convergent for $\mu>\frac{5}{2}$. Hence

$$\iint fg_\mu\,dxdy=\Sigma c_n d_\mu \alpha_\mu(\lambda_n\rho), \quad (27)$$

and our lemma follows by making $\rho\rightarrow 0$.

COR. If $\Sigma c_n d_n$ *is convergent then* $\iint fg\,dx\,dy=\Sigma c_n d_n$.

LEMMA 3. *If $f(x,y)\ \varepsilon\ L$ then $\Sigma c_n\omega_n\alpha_\mu(\lambda_n\rho)$ will be absolutely and uniformly convergent for $\mu>11/2$ and where $f(x,y)$ is continuous*

$$\lim_{\rho\rightarrow 0} f_\mu(\rho;x,y)=\sum_{\substack{\lim\\ \rho=0}} c_n\omega_n\alpha_\mu(\lambda_n\rho)=f(x,y).$$

PROOF. $$c_n=\iint f\omega_n\,dx\,dy$$

$$=O(\mu_n)\,O\left(\iint|f|\,dx\,dy\right)=O(\mu_n). \quad (28)$$

Hence

$$\Sigma\left|c_n\omega_n\alpha_\mu(\lambda_n\rho)\right|=\Sigma O(\mu_n^2)\,O(1/\lambda_n^{\mu+\frac{1}{2}})$$
$$=O(1/n^{\frac{\mu}{2}-\frac{7}{4}})=O(1),$$

provided $\mu>11/2$. To prove the second part we take $g(\xi,\eta)$ as in (24) and apply the corollary to Lemma 2.

§1.3. These three lemmas together with the known fact that $\omega_n(x, y)$ form a closed and complete set* in the spaces L^p and C^* are sufficient to prove some of the most important theorems on the summability of Fourier series of $f(x, y)$, which are necessary for us.

THEOREM 1. *If $f(x, y) \varepsilon L$, then the series $\Sigma c_n \omega_n e^{-\mu_n t}$ is absolutely and uniformly convergent for $t > 0$, and*

$$\lim_{t=0} \Sigma c_n \omega_n e^{-\mu_n t} = f(x, y)$$

at every point where $f(x, y)$ is continuous—in fact uniformly in any region in which $f(x, y)$ is continuous.

PROOF. The absolute convergence of the series for $t > 0$ follows from (28), (12) and (13). To prove the second part, we use Lemma 3. Multiplying (25) by

$$\frac{2}{(4t)^{\mu+1}\Gamma(\mu+1)}\rho^{2\mu+1}e^{-\rho^2/4t}$$

and integrating term by term with respect to ρ from 0 to ∞ and then letting $t \to 0$, we observe that

$$\lim_{t=0} \frac{2}{(4t)^{\mu+1}\Gamma(\mu+1)}\int_0^\infty \rho^{2\mu+1}e^{-\rho^2/4t} f_\mu(\rho; x, y)\, d\rho$$
$$= \lim_{t=0} \Sigma c_n \omega_n e^{-\mu_n t} = f(x, y). \quad (29)$$

In a region where $f(x, y)$ is continnous, $f_\mu(\rho; x, y) \to f(x, y)$ uniformly and hence (29) also holds good uniformly. The following is an immediate consequeince of Theorem 1.

THEOREM 2. *If $f(x, y) \varepsilon B$ or L^p, $p \geqslant 1$ then (29) holds good almost everywhere in G. But if $f(x, y) \varepsilon C^*$ then (29) holds good uniformly in $G+\Gamma$.*

§1.4. We shall now consider the determination of the necessary and sufficient conditions for $\Sigma c_n \omega_n$ to be the Fourier series of $f(x, y)$ belonging to L^p, B or C.* Given $\Sigma c_n \omega_n$ we assume that the series $\Sigma c_n \omega_n e^{-\mu_n t}$ is absolutely convergent for $t > 0$ and write

$$F(x, y; t) = \Sigma c_n \omega_n e^{-\mu_n t}. \quad (30)$$

*Cf. E4.

THEOREM 3. *A necessary and sufficient condition that $\Sigma c_n \omega_n$ may be the Fourier series of a function $f(x, y) \varepsilon L^p, p > 1$ or B, is that for all t*

$$\| F(x, y; t) \| = O(1). \tag{31}$$

PROOF. The condition is certainly sufficient. For, the space $L^p (p > 1)$ or B is weakly compact, so that if (31) is fulfilled, there is a function $f(x, y)$ belonging to the corresponding space and a decreasing sequence of positive numbers $t_n \rightarrow 0$, so that $F(x, y; t_n) \rightarrow f(x, y)$ weakly, as $n \rightarrow \infty$. Hence

$$\lim_{n=\infty} \iint F(x, y;\ t_n) \omega_\nu(x, y) = \iint f \omega_\nu dx\, dy \tag{32}$$

so that $\Sigma c_n \omega_n$ is the Fourier series of $f(x, y)$. To prove the necessity, we observe that if A denotes one of the spaces $L^p (p > 1)$, or $B, F(x, y, t)$ is a linear transformation of $f(x, y)$ in the space A. Hence $\|F(x, y; t)\| \leqslant k(t) \| f \|$, where $k(t)$ is independent of f. To prove that $k(t) = O(1)$ for all t, we take a function $f \varepsilon A$ and vanishing on a boundary strip of G. Then, for $\mu > 5/2$

$$\| f_\mu(\rho; x, y) \| \leqslant \tau(\rho) \| f \|, \tag{33}$$

where $\tau(\rho)$ is bounded at ∞ and as $\rho \rightarrow 0$, $\tau(\rho) \leqslant 1$ from (25). Therefore there is a constant τ independent of f and ρ, so that

$$\| f_\mu(\rho;\ x, y) \| \leqslant \tau \| f \|$$

and hence by (29) there is a constant k independent of f and ρ so that

$$\| F(x, y;\ t) \| \leqslant k \| f \|. \tag{34}$$

THEOREM 4. *A necessary and sufficient condition that $\Sigma c_n \omega_n$ may be the Fourier series of a function $f(x, y) \varepsilon L^p$, $1 \leqslant p < \infty$ or C^* is that*

$$\lim_{t \rightarrow 0} \| F(x, y; t) - f(x, y) \| = 0.$$

PROOF. If $f \varepsilon L$ or C, the condition (31) can be seen to be necessary but not sufficient. However, Theorem 4 holds good in this case. Let A denote one of the spaces

$$L^p (1 \leqslant p < \infty),\ C,$$

Since ω_n form a closed set in the space A, we can determine a sequence of polynomials $g_n(x, y)$, linear in ω_ν, $\nu = 1, 2, \ldots, n$ so that $\lim_{n\to\infty} \|f - g_n\| = 0$. Now

$$F(x, y; t) - f = F(x, y, t) - G_n(x, y, t)$$
$$+ G_n(x, y\, t) - g_n(x, y) + g_n(x, y) - f$$

so that

$$\|F - f\| \leqslant \|F - G_n\| + \|G_n - g_n\| + \|g_n - f\|.$$

Letting $t \to 0$ first and $n \to \infty$ next, we prove the necessity of the condition in Theorem 4. The sufficiency follows from the fact that the condition implies

$$\lim_{t\to 0} \iint F\omega_n dxdy = \iint f\omega_n dxdy = c_n.$$

THEOREM 5. *A necessary and sufficient condition that* $\Sigma c_n \omega_n$ *may be the Fourier series of a function* $f\ (x, y) \varepsilon\ C^*$ *or* L^p, $1 \leqslant p < \infty$, *is that*

$$\lim_{t,\, t' \to 0} \|F(x, y; t) - F(x, y, t')\| = 0.$$

This is an easy consequence of Theorem 4.

II. THE LINEAR HOMOGENEOUS EQUATION.

§2. Consider the homogeneous equation

$$\triangle u - \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = 0,$$

$$u(x, y, t) = 0 \text{ on } \Gamma \text{ for } t > 0, \tag{35}$$

$$u(x, y, 0) = u_0(x, y) \sim \Sigma c_n \omega_n. \tag{35'}$$

The solution of this equation, if it exists, is given by

$$u(x, y, t) = \Sigma c_n \omega_n e^{-\mu_n t}. \tag{36}$$

(1) Let $u_0(x, y)\ \varepsilon\ C^*$. Then the series on the right of (36) converges absolutely and uniformly for $t > 0$ and

$$\lim_{t\to 0} u(x, y, t) = u_0(x, y)$$

uniformly in $G + \Gamma$. This follows from Theorem 2 and further $u(x, y, t)$ is a solution of (8).

(2) If $u_0(x, y)\ \ \varepsilon\ B$ or L^p, $p \geqslant 1$, then $u(x, y, t)$ will still represent a solution of (8) fulfilling the condition (35), but (35') is satisfied in the sense that

$$\lim_{t\to 0} u(x, y, t) = u_0(x, y)$$

almost everywhere in G. This again follows from Th. 2. If $u_0(x, y) \in L^p$, $p \geqslant 1$ we can further assert by Th. 4, that

$$\lim_{t \to 0} \|u(x, y, t) - u_0(x, y)\| = 0.$$

We shall now show that if $u_0(x, y) \in C$, B or L^p, $p \geqslant 1$

$$\left.\begin{array}{l} \|\partial u/\partial x\| \\ \|\partial u/\partial y\| \end{array}\right\} = O(t^{-\frac{1}{2}}) \text{ as } t \to 0. \tag{37}$$

If $t > 0$, the series for $\partial u/\partial x$ and $\partial u/\partial y$ will be absolutely convergent on account of the order relations (14) and (28). Now if $f(x, y)$ is a function vanishing on a '*boundary strip*', then

$$f_1(\rho, x, y) = \frac{1}{\pi\rho^2} \iint_{C_\rho} f(\xi, \eta)\, d\xi\, d\eta \tag{38}$$

and

$$\frac{\partial}{\partial x} f_1(\rho; x, y) = \frac{1}{\pi\rho} \int_0^{2\pi} f(x + \rho \cos\theta, y \mp \rho \sin\theta) \cos\theta\, d\theta \tag{39}$$

so that if $\|f\| \leqslant M$, then $\|\partial f_1/\partial x\| \leqslant 2M/\rho$. A fortiori,

$$\|\partial f_\mu/\partial x\| \leqslant 2M/\rho. \tag{40}$$

If μ is large enough, the series for $\partial f_\mu/\partial x$ will be absolutely convergent for all $\rho > 0$, and hence as in (29)

$$\|\partial F/\partial x\| = O(t^{-\frac{1}{2}}) \text{ and similarly, } \|\partial F/\partial y\| = O(t^{-\frac{1}{2}}).$$

Now if $u_0(x, y) \in C^*$, B or L^p, $p \geqslant 1$ we take a sequence of functions $f_n(x, y)$ belonging to the same class as u_0 but vanishing on '*boundary strips*' so that

$$\lim_{n = \infty} \|f_n - u_0\| = 0. \tag{41}$$

It will follow from the above and (50) that

$$\|\partial u/\partial x\| \leqslant \|\partial u/\partial x - \partial F_n/\partial x\| + \|\partial F_n/\partial x\| = O(t^{-\frac{1}{2}})$$

and similarly $\qquad \|\partial u/\partial y\| = O(t^{-\frac{1}{2}})$.

(3) If $u_0(x, y) \in H_\alpha$ or H_α^p, we assume $u_0(x, y)$ to be such that we can find functions $f_n(x, y) \in H_\alpha$ or H_α^p and vanishing on *boundary strips* such that (41) is fulfilled. If $f(x, y)$ vanishes on a "*boundary strip*" belonging to H_α or H_α^p, it will follow from (39), when $\|f\| \leqslant M$, that

$$\|\partial f_1/\partial x\| = O(\rho^{1-\alpha}) \text{ so that } \|\partial f_\mu/\partial x\| = O(\rho^{1-\alpha}).$$

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Therefore arguing as before we note that

$$||\partial F/\partial x|| = O(t^{\frac{\alpha-1}{2}})$$

and hence

$$\left.\begin{matrix}||\partial u/\partial x\,||\\ ||\partial u/\partial y||\end{matrix}\right\} = O(t^{\frac{\alpha-1}{2}}).$$

III. THE LINEAR NON-HOMOGENEOUS EQUATION

§3. Let us now study the equation

$$\Delta u - \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = f(x, y, t)$$

for the boundary conditions (35) and

$$u(x, y, 0) = u_0(x, y) \equiv 0.$$

Let us suppose

$$(x\;, y, t) \sim \sum_{n=1}^{\infty} f_n(t)\omega_n(x, y) \tag{42}$$

$$u(x, y, t) \sim \Sigma u_n(t)\,\omega_n(x, y). \tag{43}$$

We then have as in P.1

$$u_n(t) = -\int_0^t e^{-\mu_n(t-s)} f_n(s)\,ds. \tag{44}$$

(i) Let $f(x, y, t)$ be a bounded measurable function vanishing identically for $t = 0$ and let $|f(x, y, t)| \leqslant M$ for $0 \leqslant t \leqslant T < \infty$. We shall now discuss the behaviour of the series on the right of (43). If we set

$$u_n^\delta(t) = -\int_0^{t-\delta} f_n(s)\, e^{-\mu_n(t-s)}\, ds$$

and $\sum u_n^\delta(t) \dfrac{\partial \omega_n}{\partial x}$

$$= -\sum \int_0^t e^{-\mu_n(t-s)} f_n(s) \frac{\partial \omega_n}{\partial x}\, ds$$

$$= -\int_0^t \sum e^{-\mu_n(t-s)} f_n(s) \frac{\partial \omega_n}{\partial x}\, ds, \tag{45}$$

the series on the right converges absolutely and therefore the series on the left converges absolutely. Hence the function on the left of (45) is a continuous function of

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(x, y, t) in the cylinder S whose base is G and whose axis is parallel to 't' axis, with $0 \leqslant t \leqslant T$. If $0 < \delta < \delta' < T/2$,

$$\Sigma u_n^{\delta} \frac{\partial \omega_n}{\partial x} - \Sigma u_n^{\delta'} \frac{\partial \omega_n}{\partial x} = \int_{t-\delta}^{t-\delta'} \Sigma e^{-\mu_n(t-s)} f_n \frac{\partial \omega_n}{\partial x} ds = O\left\{\int_{t-\delta}^{t-\delta'} \frac{ds}{(t-s)^{\frac{1}{2}}}\right\}$$
$$= O\left(\delta'^{\frac{1}{2}} - \delta^{\frac{1}{2}}\right), \tag{46}$$

the constant in O being $\leqslant 2M$. Therefore

$$\left| \Sigma u_n^{\delta} \frac{\partial \omega_n}{\partial x} - \Sigma u_n^{\delta'} \frac{\partial \omega_n}{\partial x} \right| \leqslant 2M\left(\delta'^{\frac{1}{2}} - \delta^{\frac{1}{2}}\right) \to 0 \tag{47}$$

as $\delta, \delta' \to 0$. Further

$$\left| \Sigma e^{-\mu_n \delta} u_{n(t)\omega_n} \right| = \left| \int_0^t \Sigma e^{-\mu_n(t-s+\delta)} f_{n\omega_n} ds \right| \leqslant Mt \tag{48}$$

and

$$\left| \Sigma e^{-\mu_n \delta} u_{n(t)} \frac{\partial \omega_n}{\partial x} \right| = \left| \int_0^t \Sigma e^{-\mu_n(t-s+\delta)} f_n \frac{\partial \omega_n}{\partial x} ds \right|$$
$$\leqslant M \int_0^t \frac{ds}{\sqrt{t+\delta-s}} \leqslant 2M(\sqrt{t+\delta} + \sqrt{\delta}). \tag{49}$$

Now, while (48) shows that the series on the right of (43), where $u_n(t)$ is given by (44), is the Fourier series of $u(x, y, t)$, (47) and (49) show that $\partial u/\partial x$ exists and is continuous in S. It can similarly be shown that $\partial u/\partial y$ is continuous. A fortiori $u(x, y, t)$ is continuous in S. In as far as $\partial u/\partial x$ and $\partial u/\partial y$ are continuous, $u(x, y, t)$ is Lipschitzian with respect to x and y and hence* its Fourier series is uniformly convergent in S. Considered as a function of t, we observe that,

$$u(x, y, t+\eta) - u(x, y, t)$$
$$= -\int_t^{t+\eta} \Sigma f_{n\omega_n} e^{-\mu_n(t+\eta-s)} ds + \int_0^t \Sigma f_{n\omega_n} e^{-\mu_n(t-s)} \left(1 - e^{-\mu_n \eta}\right) ds$$
$$= -\int_t^{t+\eta} \Sigma f_{u\omega_n} e^{-\mu_n(t+\eta-s)} ds + \eta \int_0^t \Sigma f_{n\omega_n \mu_n} e^{-\mu_n(t+\theta\eta-s)} ds, \tag{50}$$

where $0 < \theta < 1$ by the mean-value theorem.

The first series on the right of (50) is $O(1)$ by Theorem 3 while the second is $O\left(\frac{1}{t+\vartheta\eta-s}\right)$ by a further differentiation of (37).

* *Cf.* E 2.

Hence

$$|u(x, y, t+\eta)-u(x, y, t)|$$
$$= O\left(\int_t^{t+\eta} ds\right)+O\left(\eta\int_0^t \frac{ds}{t+\vartheta\eta-s}\right)= O(\eta)+O\left(\eta \log \frac{\vartheta\eta}{t+\vartheta\eta}\right)$$
$$= O(\eta \log \eta), \text{ as } \eta \to 0. \quad (51)$$

The derivatives $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial u}{\partial t}$ will not in general exist. We can however give a significance to the operator $\delta u = \triangle u - \frac{\partial u}{\partial t}$ as we did in P.1 and assert there *'exists'* a solution of (9) in a general sense. In fact let s be a small cylinder contained in S with its axis parallel to that of S and containing the point (x, y) at its centre. Let the projection of the base of s in the (x, y) plane be the region g with boundary γ. We then define

$$\delta u = \lim_{|s|=0} \frac{1}{|s|}\iint_\sigma \left(u d\xi d\eta + \frac{\partial u}{\partial n} d\gamma dt\right), \tag{52}$$

where $d\xi d\eta$ is an areal element in g, $d\gamma$ is a line element in γ and σ is the boundary of s and $\partial u/\partial n$ is the derivative of u along the exterior normal to γ. As in P.1 it can be shown that, if $f(x, y, t)$ is bounded and measurable, (52) exists almost everywhere, and that it exists everywhere if $f(x, y, t)$ is continuous in S.

(ii) However, if $f(x, y, t)$ belongs to H_α for every t, we can show that $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial u}{\partial t}$ exist and are continuous using the same argument as in (47).

§3.1. Let $\rho(x, y, t)$ be a function defined in S and u a solution of the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial t} = \rho(x, y, t) \tag{53}$$

for the boundary conditions (35) and (35′). If ρ belongs to some space, then u, $\partial u/\partial x$, $\partial u/\partial y$ belong to some corresponding spaces, and the relation between ρ and these functions will be defined by linear transformations. We have in fact the following

THEOREM 6. *In the space B of bounded measurable functions defined in S, $u(x, y, t)$ is a completely continuous linear transformation of ρ, and $\partial u/\partial x$, $\partial u/\partial y$ are linear transformations of ρ.*

PROOF. We write

$$u = L(\rho), \frac{\partial u}{\partial x} = L_p(\rho), \frac{\partial u}{\partial y} = L_q(\rho).$$

That these are linear transformations in B is evident from (44), (48) and (49). That $L(\rho)$ is a completely continuous transformation follows from the fact that $u\,(x, y, t)$ is Lipschitzian with respect to x and y and satisfies the relation (51) as regards the variable t. In fact this means that every bounded set is transformed into a bounded set of equi-continuous functions, which is a compact set. In as far as a continuous transformation is completely continuous if it transforms bounded sets into compact sets, the proof of the theorem is complete.

We have now only to apply Fixpunkt-Satz of Schauder in order to obtain the existence theorems for the non-linear equation. We propose to obtain these results in the next part.

IV. THE NON-LINEAR EQUATION.

§ 4. Consider the non-linear equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = f(x, y, t, u) \tag{54}$$

for the boundary conditions (35) and (35′).

Let $f(x, y, t, u)$ be a bounded measurable function of all the variables and uniformly continuous with respect to u. Then $f\Big(x, y, t, L(\rho)\Big) = F(\rho)$ is a completely continuous transformation of the space B into itself. Therefore there exists an invariant point in B, i.e. there is a function $\rho(x, t) \in B$ so that $\rho = F(\rho)$. This means that there is a solution of (54). If we assume however that

$$|f(x, y, t, u)| \leqslant \phi(t), \ \ \phi(0) = 0, \tag{55}$$

where $\phi(t)$ is a positive non-decreasing function of t, then the convex closed sphere

$$\|\rho\| \leqslant M \tag{56}$$

is transformed into the sphere $\|F(\rho)\| \leqslant \phi(t)$ and $\phi(t) < M$ for small values of t. For these values of t the sphere (56) is transformed into itself and we have an invariant point. By choosing M large enough, there exist solutions for all finite t. Hence, we have

THEOREM 7. *If the function on the right of* (54) *is a bounded function of all the variables, uniformly continuous with respect to u and fulfilling the inequality* (55), *then there is a solution of* (54) *for the boundary conditions* (35) *and* (35′) *for all finite t.*

§4.1. We will now consider the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = f(x, y, t, u, p, q), \tag{57}$$

where $p = \partial u/\partial x$, $q = \partial u/\partial y$, for the boundary conditions (35) and (35′). Let

(*a*) $f(x, y, t, u, p, q)$ be a continuous function of all the variables vanishing on the boundary of G and

(*b*) $|f(x, y, t, u, p, q)| \leqslant \phi(t)$, $\phi(0) = 0$, where $\phi(t)$ is a non-decreasing positive function of t. Then

$$f[x, y, t, L(\rho), L_p(\rho), L_q(\rho)] = F(\rho)$$

is a continuous transformation of C into itself whenever $\rho \varepsilon C$. As before, the sphere $\|\rho\| \leqslant M$ is transformed into itself, if t is small enough, and as C is separable, there is an invariant point. That is, there is a solution of (57). As before, by taking M large enough we see there exist solutions for all finite t. Hence we have

THEOREM 8. *The equation* (57) *has a solution for all finite t and for boundary conditions* (35) *and* (35′) *provided the function on the right of* (57) *fulfils the conditions* (*a*) *and* (*b*).

ON A CONGRUENCE PROPERTY OF THE DIVISOR FUNCTION I*

BY

L. G. SATHE, *Calcutta.*

[Received in revised form, 6 September, 1943.]

In my attempt to generalize the result of Dr. S. S. Pillai "On a congruence property of the divisor function"† to composite numbers, I found his formula

$$N(k, x) = \sum_{q,\, r} (-1)^t F(q^{rk-1})$$ to be incorrect and hence

his final result is not true. I prove that

$$N(k, x) = A_k x + O(x^{1/(k-1)} \log x),$$

where k is a prime $\geqslant 3$ and

$$A_k = 1 - \prod_p \{ 1 - (p-1)^k/(p^k - 1) \},$$

p running through all primes $\geqslant 2$; $N(k, x)$ being the number of $n \leqslant x$ for which $d(n)$, the number of divisors of n, is a multiple of k.

Let $f(n)$ denote the number of different prime factors of n, $t(n) = f(n) - 1$, $\phi(n) =$ Euler's function, $\nu = 1/(k-1)$. q denotes a number of the form $p_1^{r_1 k-1}\ p_2^{r_2 k-1} \ldots p_\mu^{r_\mu k-1}$, and is not the same as Pillai's q, $r_1, r_2, \ldots$ being positive integers $\geqslant 1$. $F(m) = F(m, x)$ denotes the number of $n \leqslant x$ which is a multiple of m and is such that $(m, n/m) = 1$. $G(m, x)$ denotes the number of $n \leqslant x$ and prime to m.

LEMMA 1. $N(k, x) = \sum_{q \leqslant x} (-1)^{t(q)} F(q)$.

* I wish to thank Prof. F.W. Levi, and Dr. S. S. Pillai at whose suggestion this problem was taken up.

† *Jour. Ind. Math. Soc.* (2) 6 (1942).

Let $n = p_1^{a_1} \ldots p_\lambda^{a_\lambda}$ belong to $N(k, x)$ and g be the number of a indices $\equiv -1 \pmod{k}$. Then $g \geqslant 1$ since n belongs to $N(k, x)$. The number of times n is counted in the above formula $= \binom{g}{1} - \binom{g}{2} + \ldots + (-1)^{g-1}\binom{g}{g} = 1$. Moreover n is not counted at all if it does not belong to $N(k, x)$. Hence the result.

LEMMA 2. $\sum_{q \leqslant x} 2^{f(q)} = O(x^\nu \log x)$.

Let $s = p_1^{\gamma_1} \ldots p_\mu^{\gamma_\mu}$, then $s^{k-1} \leqslant q \leqslant x$. Also $f(q) = f(s)$ and two numbers s are equal if and only if the corresponding numbers q are equal. In other words there is a one-one correspondence between the q's and s's. Hence,

$$\sum_{q \leqslant x} 2^{f(q)} \leqslant \sum_{s \leqslant x^\nu} 2^{f(s)} = O \sum_{n \leqslant x^\nu} d(n) = O(x^\nu \log x).*$$

It may be noted that the number $M(x)$ of $q \leqslant x$ is not greater than x^ν.

LEMMA 3. $\sum_{q > x} \frac{1}{q} = O(x^{\nu-1})$.

This lemma is due to Pillai.

Let $M(x)$ denote the number of $q \leqslant x$ and $m = [x]$. Then

$$\sum_{q > x} 1/q = \sum_{n > m} \{ M(n) - M(n-1) \}/n$$

$$= -\frac{M(m)}{m} + \sum_{n \geqslant m} M(n) \left\{ \frac{1}{n} - \frac{1}{n+1} \right\}$$

$$= -\frac{M(m)}{m} + O \sum_{n \geqslant m} \frac{M(n)}{n^2} = O(m^{\nu-1}) + O \sum_{n \geqslant m} 1/n^{2-\nu}$$

$$= O(m^{\nu-1}) = O(x^{\nu-1}).$$

THEOREM. $N(k, x) = Ax + O(x^\nu \log x)$ *if* $k \geqslant 3$.

* Hardy and Weight, *Introduction to the Theory of Numbers* p. 262.

Pillai* has proved that

$$G(m, x) = x.\frac{\phi(m)}{m} + O(2^{f(m)}).$$

Also

$$F(q) = G(q, x/q).$$

Therefore,

$$\mathcal{N}(k, x) = x \sum_{q \leqslant x} (-1)^{t(q)}\phi(q)/q^2 + O \sum_{q \leqslant x} 2^{f(q)}$$

$$= x\sum_{q}(-1)^{t(q)}\phi(q)/q^2 + O(x) \sum_{q>x}\frac{1}{q} + O \sum_{q \leqslant x} 2^{f(q)}.$$

The series $A = \sum_{q} (-1)^{t(q)}\phi(q)/q^2$ converges absolutely when $k \geqslant 3$ since $A < \Sigma 1/q < \Sigma 1/s^{k-1} \leqslant \Sigma 1/n^{k-1}$.

Hence when $k \geqslant 3$, $\mathcal{N}(k, x) = Ax + O(x^{\nu} \log x)$. Now

$$\prod_{p \leqslant x} \left\{ 1-(p-1)/(p^k-1) \right\}$$

$$= \prod_{p \leqslant x} \left\{ 1-(p-1)\left(\frac{1}{p^k}+\frac{1}{p^{2k}}+\dots\right)\right\}$$

$$= 1 + \sum_{q \leqslant x} (-1)^{f(q)}\phi(q)/q^2 + \sum_{q>x}{}' (-1)^{f(q)}\phi(q)/q^2$$

$$= 1 - \sum_{q \leqslant x} (-1)^{t(q)}\phi(q)/q^2 + \sum_{q>x}{}' (-1)^{f(q)}\phi(q)/q^2,$$

where Σ' is taken over those $q > x$ whose prime factors are all $\leqslant x$. The modulus of the last term on the right $\leqslant \sum_{q>x} 1/q = O(x^{\nu-1})$. Hence,

$$\prod_{p} \left\{ 1-(p-1)/(p^k-1) \right\} = 1 - \sum_{q} (-1)^{t(q)}\phi(q)/q^2 = 1-A.$$

The theorem in question follows. Also, since

$$\prod_{p}\left\{ 1-(p-1)/(p^k-1) \right\} = \prod_{p}(1-1/p^{k-1})/\prod_{p}(1-1/p^k)$$

$$= \zeta(k)/\zeta(k-1),$$

$$A = 1-\zeta(k)/\zeta(k-1).$$

* *loc. cit.*

ON A CONGRUENCE PROPERTY OF THE DIVISOR FUNCTION II.

BY

L. G. SATHE, *Calcutta.*

[Received 24 November 1943].

1. Let $d(n)$ be the number of divisors of n and $N(k, x)$ be the number of $n \leqslant x$ for which $d(n) \equiv 0 \pmod{k}$. Then in I* it is proved that $N(k, x) \sim [1 - \zeta(k)/\zeta(k-1)]x$ when k is a prime. In this paper I prove

THEOREM 1. *$N(k, x) = Cx + O(x^{1/(m_1-1)+\varepsilon})$, for every $\varepsilon > 0$ where $C = C(k) > 0$ is a constant and $k = m_1 m_2$; $m_1 < m_2$; m_1 and m_2 being different odd primes.*

THEOREM 2. *$N(2k, x) \sim N(k, x)$ when k is odd.*

THEOREM 3. *$N(k, x) \neq o(x)$ for any k.*

NOTATION. $f(n)$ = the number of different prime factors of n; $t(n) = f(n) - 1$; $\phi(n)$ is Euler's function; $F(m, x)$ denotes the number of $n \leqslant x$ which is a multiple of m such that $(m, n/m) = 1$; u, v, w are sets of integers defined by the formulae $u = p_1^{a_1 m_1 - 1} \dots p_r^{a_r m_1 - 1}$, $r \geqslant 1$, $a_i \not\equiv 0 \pmod{m_2}$, $a_i \geqslant 1$; $v = q_1^{b_1 m_2 - 1} \dots q_s^{b_s m_2 - 1}$, $s \geqslant 1$, $b_i \not\equiv 0 \pmod{m_1}$, $b_i \geqslant 1$; $w = \gamma_1^{c_1 m_1 m_2 - 1} \dots r_t^{c_t m_1 m_2 - 1}$, $t \geqslant 1$; $c_i \geqslant 1$, p, q, r denoting different primes. $M_1(x)$, $M_2(x)$, $M_3(x)$, $M_4(x)$, $M_5(x)$ denote respectively the number of u, v, w, uv and uvw less than or equal to x.

2. LEMMA 1.

$$N(k, x) = \sum_{\substack{uv \leqslant x \\ (u,v)=1}} (-1)^{t(u)+t(v)} F(uv, x) + \sum_{w \leqslant x} (-1)^{t(w)} F(w, x) - \sum_{uvw \leqslant x} (-1)^{t(u)+t(v)+t(w)} F(uvw, x),$$

* *Jour. Ind. Math. Soc.* (2) 7 (1943), 143-5.

the last summation being extended over those u, v, w which are relatively prime.

It is obvious that $d(uv)$, $d(w)$, $d(uvw)$ are all multiples of k for every integer of those sets. So the contribution to the above sum by every n for which $d(n) \not\equiv 0 \pmod{k}$ is equal to zero. It is therefore sufficient to prove that every n for which $d(n)$ is a multiple of k contributes just one to the above sum.

Let n be a fixed integer $\leqslant x$ such that $d(n) \equiv 0 \pmod{k}$. Then n has the form $n = p_1^{a_1 m_1 - 1} \dots p_\lambda^{a_\lambda m_1 - 1} q_1^{b_1 m_2 - 1} \dots q_\mu^{b_\mu m_2 - 1} r_1^{c_1 m_1 m_2 - 1} \dots r_\nu^{c_\nu m_1 m_2 - 1} l_1^{\alpha_1} \dots l_t^{\alpha_t}$ where a, b, c satisfy the same conditions as laid down in the definitions of u, v and w; $\alpha_i \geqslant 0$ is not a multiple of m_1 or m_2 (except when it is zero). The l's are different primes.

(1) Suppose that $\nu = 0$. Then $\lambda \neq 0$, $\mu \neq 0$.

In this case there is no w or uvw of which n can be a multiple so that the last two sums on the right side are each equal to zero. To calculate the first sum, consider a certain uv of which n is a multiple. Let λ' and μ' be the different prime factors of u and v respectively. Then the contribution by it is $(-1)^{\lambda'-1} . (-1)^{\mu'-1}$.

But there are exactly $\binom{\lambda}{\lambda'}\binom{\mu}{\mu'}$ integers uv which have λ' and μ' different prime factors and of which n is a multiple.

Hence the total contribution to the first sum by n is

$$\sum_{\lambda'=1}^{\lambda} \sum_{\mu'=1}^{\mu} (-1)^{\lambda'-1} . (-1)^{\mu'-1} \binom{\lambda}{\lambda'}\binom{\mu}{\mu'}$$

$$= \sum_{\lambda'=1}^{\lambda} (-)^{\lambda'-1} \binom{\lambda}{\lambda'} \sum_{\mu'=1}^{\mu} (-1)^{\mu'-1} \binom{\mu}{\mu'}$$

$$= \sum_{\lambda'=1}^{\lambda} (-1)^{\lambda'-1} \binom{\lambda}{\lambda'} . 1 = 1.$$

(2) Suppose now that $\nu \neq 0$ but one or both of λ and μ are equal to zero. Then only w can enter the summation and as before the total contribution by n is

$$\sum_{\nu'=1}^{\nu} (-1)^{\nu'-1} \binom{\nu}{\nu'} = 1.$$

(3) Finally consider the case in which none of λ, μ, ν are equal to zero Then, as before, the contribution by n to the first two summations is each equal to 1. Also arguing as in (1) we see that the contribution by n to the last sum is

$$\sum_{\lambda'=1}^{\lambda} (-1)^{\lambda'-1} \binom{\lambda}{\lambda'} \sum_{\mu'=1}^{\mu} (-1)^{\mu'-1} \binom{\mu}{\mu'} \sum_{\nu'=1}^{\nu} (-1)^{\nu'-1} \binom{\nu}{\nu'} = 1.$$

Hence the total contribution $= 1+1-1 = 1$; and the lemma follows since these are the only possible cases.

Lemma 2. *$M_4(x)$ and $M_5(x)$ are each equal to $O(x^{1/(m_1-1)})$.*

If $u = p_1^{a_1 m_1 - 1} \ldots p_\gamma^{a_\gamma m_1 - 1} \leqslant x$, and $u' = p_1^{a_1} \ldots p_\gamma^{a_\gamma}$ then $u'^{(m_1-1)} \leqslant u \leqslant x$.

Also two integers of the set u' are equal if and only if the corresponding u's are identical. Hence the number of $u \leqslant x$ is not greater than the number of $u' \leqslant x^{1/(m_1-1)}$.

Hence $M_1(x) \leqslant x^{1/(m_1-1)}$.

Similarly $M_2(x) \leqslant x^{1/(m_2-1)}$, $M_3(x) \leqslant x^{1/(k-1)}$.

Also $M_4(x) = \sum_{v \leqslant x} M_1(x/v) \leqslant x^{1/(m_1-1)} \sum_{v \leqslant x} 1/v^{1/(m_1-1)}$

$$= x^{1/(m_1-1)} \sum_{n=1}^{x} \frac{M_2(n) - M_2(n-1)}{n^{1/(m_1-1)}}$$

$$\leqslant x^{1/(m_1-1} \sum_{n=1}^{\infty} M_2(n) \left\{ \frac{1}{n^{1/(m_1-1)}} - \frac{1}{(n+1)^{1/(m_1-1)}} \right\}.$$

Now the general term in the above series is

$$= O\left\{ \frac{1}{n^{1/(m_1-1)-1/(m_2-1)+1}} \right\}$$

so that the series is convergent. Hence

$$M_4(x) = O(x^{1/(m_1-1)}).$$

It follows similarly that $M_5(x) = O(x^{1/(m_1-1)})$.

LEMMA 3. *The series* $\sum\limits_{uv} \frac{1}{uv}$, $\sum\limits_{w} \frac{1}{w}$, $\sum\limits_{uvw}^{1/uvw} \frac{1}{uvw}$ *are all convergent if* $m_1 \geqslant 3$.

Generally if q is any set of positive integers and $M(x)$ the number of $q \leqslant x$, then the series $\Sigma 1/q$ converges if $M(x) = O(x^{\alpha})$, where $\alpha < 1$.

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$$\text{For } \sum \frac{1}{q} = \sum_{n=1}^{\infty} \frac{M(n)-M(n-1)}{n}$$

$$= -M(1) + \sum_{1}^{\infty} M(n) \left\{ \frac{1}{n} - \frac{1}{n+1} \right\} = \sum \frac{1}{n^{2-\alpha}} = O(1).$$

Hence from Lemma 2 the required result follows.

LEMMA 4. $\sum\limits_{uv>x} \frac{1}{uv}$, $\sum\limits_{w>x} \frac{1}{w}$, $\sum\limits_{uvw<x} \frac{1}{uvw}$ *are each equal to* $O(x^{-1+1/(m_1-1)})$.

We have

$$\sum_{uv>x} \frac{1}{uv} = \sum_{n>x} \frac{M_4(n)-M_4(n-1)}{n}$$

$$= -\frac{M_4(t)}{t+1} + \sum_{n=t+1}^{\infty} M_4(n)\left[\frac{1}{n} - \frac{1}{n+1}\right],$$

where $t = [x]$. From Lemma 2 the result follows. Similarly for the other sums.

LEMMA 5. $\sum\limits_{uv \leqslant x} 2^{f(uv)} = O(x^{1/(m_1-1)+\varepsilon})$ *for every* $\varepsilon > 0$.

$$\text{For } \sum_{uv \leqslant x} 2^{f(uv)} \leqslant \sum_{uv \leqslant x} d(uv) = O\Big(x^{\varepsilon} \sum_{uv \leqslant x} 1\Big)$$

$$= O(x^{1/m_1-1)+\varepsilon}).$$

It follows similarly that

$$\sum_{uvw \leqslant x} 2^{f(uvw)} = O(x^{1/(m_1-1)+\varepsilon})$$

$$\sum_{w \leqslant x} 2^{f(w)} = O(x^{1/(k-1)+\varepsilon}) = O(x^{1/(m_2-1)+\varepsilon}).$$

3. We now proceed to the proofs of the theorems.

3.1. Proof of Theorem 1.

We know from I that $F(m, x) = x\phi(m)/m^2 + O(2^{f(m)})$.

Hence from Lemma 1 we get

$$N(k, x) = x\left\{\sum_{uv \leqslant x} (-1)^{t(u)+t(v)} \frac{\phi(uv)}{u^2v^2} + \sum_{w \leqslant x} (-1)^{t(w)} \frac{\phi(w)}{w^2} + \sum_{uvw \leqslant x} (-1)^{t(u)+t(v)+t(w)} \frac{\phi(uvw)}{u^2v^2w^2}\right\}$$

$$+ O\left\{\sum_{uv \leqslant x} 2^{f(uv)} + \sum_{w \leqslant x} 2^{f(w)} + \sum_{uvw \leqslant x} 2^{f(uvw)}\right\}$$

$$= cx + xO\left\{\sum_{uv > x} 1/uv + \sum_{w > x} 1/w + \sum_{uvw > x} 1/uvw\right\} + O(x^{1/(m_1 - 1) + \varepsilon}),$$

where

$$c = \sum_{uv} (-1)^{t(u)+t(v)} \frac{\phi(uv)}{u^2v^2} + \sum_{w} (-1)^{t(w)} \frac{\phi(w)}{w^2} - \sum_{uvw} (-1)^{t(u)+t(v)+t(w)} \frac{\phi(uvw)}{u^2v^2w^2}.$$

Each of the above series is absolutely convergent by Lemma 3. Hence by Lemma 4,

$N(k, x) = cx + O(x^{1/(m_1 - 1) + \varepsilon})$ if $m_1 \geqslant 3$. That the constant $c > 0$ follows from Theorem 3.

3.2. Proof of Theorem 2.

Plainly we have $N(k, x) = N(2k, x) + X$, where X is the number of $n \leqslant x$ for which $d(n) \equiv 0 \pmod{k}$ but $d(n) \not\equiv 0 \pmod{2k}$, being any odd integer. It follows that every number of the set X must have an odd number of divisors and hence all these numbers are perfect squares.

Therefore $X = O(\sqrt{x})$.

Hence $N(2k, x) = N(k, x) + O(\sqrt{x})$.

Since by Theorem 3, $N(k, x) \neq o(x)$ we get

$N(2k, x) \sim N(k, x)$ when k is odd.

This theorem includes the case excluded by Theorem 1. For when k is a prime, by I we get

$$\mathcal{N}(2k, x) = \{ 1 - \zeta(k) / \zeta((k-1) \} x + O(\sqrt{x}).$$

3.3. PROOF OF THEOREM 3.

We define a set of integers $q = p_1^{a_1 k-1} \ldots p^{a_\mu k-1}$; $\mu \geqslant 1$; $a_i \geqslant 1$; k being any given integer $\geqslant 3$. Consider the expression

$$E = \sum_{q \leqslant x} (-1)^{t(q)} F(q, x).$$

As in Lemma 1 proved in I every integer n which is a multiple of a q such that $(q, n/q) = 1$ contributes just one to E. Also $d(n) \equiv O(\text{mod } k)$ for all such n; and n is not counted at all if it does not satisfy these conditions.

Hence $\mathcal{N}(k, x) \geqslant E$. In I it is proved that

$$E = \{ 1 - \zeta(k)/\zeta(k-1) \} x + O(x^{1/(k-1)}) \text{ if } k \geqslant 3.$$

Also $\mathcal{N}(2, x) \sim x$. Hence the theorem follows.

INDIAN MATHEMATICAL SOCIETY

The Narasinga Rao Medal For Mathematical Research

This is a gold medal which will be awarded periodically by the Indian Mathematical Society out of the interest on an endowment fund created for the purpose by Dr. A. Narasinga Rao of the Annamalai University. The object of the award is to encourage mathematical research and to direct attention to modern topics which have not received sufficient attention in this country.

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The first problem has been selected and is given below :—

Problem

"It is required to characterise and study the strongest topology that can be imposed on the linear space of those real-number sequences $a = (a_1, a_2, a_3, \ldots)$, *that have a convergent sum* $S(a) = a_1 + a_2 + \ldots.$ *to infinity, so that* $S(a)$ *is a continuous functional of* a."

Explanation

The sought topology T must be a group-topology, so that for any two sequences a, b of the space. the sequence $a-b$ must be a continuous function of the two variables a,b. It is also clear that T must be "weaker" than the direct product topology of a sequence of real-number spaces, so that any element a_n of a will be a continous functional of a, in the topology T. It is suspected that T is not metrisable, but is a uniform-structure topology in the sense of Andre Weil.

This is, in a sense, a key-problem in the subject of linear and function-spaces. The space of real-number sequences with convergent sum, is not one of the regular kinds of spaces listed by Banach in his *Linear operators,* but is typical of the more troublesome intermediate kinds of space which arise. The solution of this problem will serve to indicate the nature of these intermediate spaces generally.

If we confine ourselves to the Lebesgue Integral, there is no exact integral analogue to the above problem in function-spaces; for, the Lebesgue integral is always absolutety convergent. We can, however, formulate an analogous problem for function-spaces with a non-absolutely convergent integral like the Denjoy integral; *viz.*

"If F is the space of all Denjoy-integrable functions defined in a finite interval (0,1), to find the strongest topology on F, for which the Denjoy-integral of any function is, a continuous functional of that function."

Remarks on this strongest topology T may be made, analogous to these already made in the case of sequences.

As regards references likely to be useful in connection with the Prize Problem, the idea of strong and weak topologies will be found referred to in Alexandroff and Hopf: *Topologie* and Banach's *Theorie des operations lineaires.*

The last date for the submission of the solutions of the Prize Problem will be the 31st July, 1945. Theses should be sent by registered post to Dr. A. Narasinga Rao, Annamalai University, Annamalainagar.

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Vol. VI (1932) p. 198, line 7 from bottom.
for 'all. . . .$r+s$' read 'all consecutive values of k for which $r+k$'

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The QUESTIONS and DISCUSSIONS, which are timely and interesting, cover a wide variety of topics.

Surveys of the contents of recent books constitute a valuable guide to current mathematical literature.

The topics in the department of Undergraduate Mathematics Clubs have excited wide interest among those engaged in teaching mathematics.

The NEWS and NOTICES cover a wide range of interest and information.

The Problems and Solutions hold the attention and activity of a large number of persons who are lovers of Mathematics for its own sake.

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